ACUTE TRIANGULATIONS

Doctoral Thesis

by

Liping Yuan

Supervised by

Prof. Dr. Tudor Zamfirescu

DEPARTMENT OF MATHEMATICS

DORTMUND UNIVERSITY

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CONTENTS

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Chapter 1. Introduction

The triangulation is a basic geometric notion. It is fundamental in problems arising in computer graphics, physical simulation, and geographical information systems. Most applications demand not just any triangulation, but rather one with triangles satisfying certain shape and size criteria. It is generally true that large angles are undesirable, and a bound of $\pi/2$ on the largest angles has special importance.

A *triangulation* of a two-dimensional space means a collection of (full) triangles covering the space, such that the intersection of any two triangles is either empty or consists of a vertex or of an edge. A triangle is called *geodesic* if all its edges are *segments*, i.e., shortest paths between the corresponding vertices. We are interested only in *geodesic triangulations*, all the members of which are, by definition, geodesic triangles. An *acute* (*non-obtuse*) triangulation is a triangulation whose triangles have all their angles less (not larger) than $\frac{\pi}{2}$.

The discussion of non-obtuse and acute triangulations has one of its origins in a problem of Stover reported in 1960 by Gardner in his Mathematical Games section of the Scientific American (see [20], [21]). There the question was raised whether a triangle with one obtuse angle can be cut into smaller triangles, all of them acute. Another, even earlier, interest in non-obtuse triangulations stems from the discretization of partial differential equations [33].

In 1960 Burago and Zalgaller [11] deeply investigated acute triangulations of polygonal complexes. However, their proofs cannot be used to obtain a good estimate of the actual number of triangles in it. What can be said about the *size*, i.e., the number of triangles, of an acute triangulation of a given polygon? In the same year, Burago and Zalgaller [11] and, independently, Manheimer [37] considered the case of obtuse triangles. In 1980 Cassidy and Lord [12] discussed the acute triangulations of squares and in 2000, Hangan, Itoh and Zamfirescu [25] considered the case of rectangles. In 2001 Maehara [34] investigated the acute triangulations of all quadrilaterals (convex or not). In Chapter 2, we discuss the acute triangulations of trapezoids and in Chapter 3 those of pentagons.

On the other hand, several heuristic methods have been developed to compute non-obtuse triangulations of polygons [3, 4, 6, 7, 38, 41]. In 1995 Bern, Mitchell and Ruppert [10] gave an algorithm for triangulating n-gons into $O(n)$ non-obtuse triangles. In 2002 Maehara [35] provided acute triangulations for any polygon on the basis of the existence of a non-obtuse triangulation. He gave the upper bound $2 \cdot 6^5 N$ for the number of triangles in the acute triangulation, where N denotes the number of triangles in the existing non-obtuse triangulation. In Chapter 4, we consider the acute triangulations of n-gons in an entirely different way, and improve Maehara's upper bound to $22N$. In addition, we obtain an upper bound for the size of the acute triangulations of n-gons depending on n.

At the same time, compact convex surfaces have also been triangulated. In 2000 Hangan, Itoh and Zamfirescu started the investigation of acute triangulations of all platonic surfaces, which are the surfaces of the five well-known Platonic solids [25, 26, 27]. But the case of arbitrary convex surfaces is very difficult, and even the case of doubly covered convex sets is still difficult. In 2004 C. Zamfirescu considered the acute triangulation of the doubly covered triangles [52]. In Chapter 5, we continue this line of research and investigate the acute triangulations of doubly covered quadrilaterals, doubly covered pentagons and doubly covered convex sets. Moreover, the acute triangulations of several well-known surfaces, such as flat tori and Möbius strips, are also considered in Chapter 5.

The finite elements method requires triangles having not too small and not too large angles, that is all angles are (uniformly) bounded away from 0 and π. Also, there is a new task to provide both acute triangles and a positive lower bound on the angles. In Chapter 6 we consider in the case of rectangles acute triangulations whose angles are bounded away from 0.

Chapter 2. Acute Triangulations of Convex Quadrilaterals

§2.1 Introduction

What can be said about the *size*, card$\mathcal{T}$ of an acute triangulation $\mathcal{T}$ of a given polygon? In 1960, Burago and Zalgaller [11] and, independently, Manheimer [37] proved that every obtuse triangle can be triangulated into 7 acute triangles, and 7 is the minimum number. In 1980, Cassidy and Lord [12] showed that every square can be triangulated into 8 acute triangles, and 8 is the minimum number. In 2000, Hangan, Itoh and Zamfirescu [25] extended this result to any rectangle.

Now, for a polygon Γ, let $f(\Gamma)$ denote the minimum size of an acute triangulation of Γ, and let $f(n)$ denote the maximum value of $f(\Gamma)$ for all n-gons Γ. Thus $f(3) = 7$. In 2001, Maehara [34] proved that $f(4) = 10$. Since the example used to prove that $f(4) \geq 10$ is a non-convex quadrilateral, Maehara [34] asked whether $f(\Gamma_4) \leq 8$ holds for every convex quadrilateral. We prove this here for the case of trapezoids.

§2.2 Preliminaries

A polygon Γ is a planar set homeomorphic to the compact disc, having as boundary $bd\Gamma$ a finite union of line-segments.

Let $\mathcal{T}$ be an acute triangulation of a polygon Γ. A vertex P of $\mathcal{T}$ is called a *corner vertex* if P is a vertex of $bd\Gamma$;

a *side vertex* if P lies on $bd\Gamma$ but is not a corner vertex;

an *interior vertex* otherwise.

We can regard $\mathcal{T}$ as a plane graph, that is, a planar graph embedded in the plane. Clearly, a side vertex has degree at least 4 and an interior vertex has degree at least 5.

We will make repeated use of the obvious

Proposition 2.2.1. *If ABC is an acute triangle and A' lies in a sufficiently small neighbourhood of A, then the triangle $A'BC$ is acute too.*

The following result of Maehara will also be used.

Proposition 2.2.2. ([34]) *Let ABC be a triangle with acute angles at B and C, and let $P \in \mathrm{relint} AC$. If the angle at A is acute (non-acute), then there is an acute triangulation $\mathcal{T}$ of ABC with size* 4 (7) *such that P is the only side vertex on AC.*

By $|s|$ we denote the length of the segment s and by $\overline{AB}$ the line passing through A and B.

§2.3 Acute Triangulations of Trapezoids

A *trapezoid* is a quadrilateral two sides of which are parallel.

Theorem 2.3.1. *Let $\mathcal{T}_4$ be the family of all trapezoids; then $f(\mathcal{T}_4) = 8$.*

Proof. By [25], any acute triangulation of a rectangle has size at least 8, whence $f(\mathcal{T}_4) \geq 8$. Now we'll prove that $f(\mathcal{T}_4) \leq 8$.

Let $ABCD$ be a trapezoid with $AD \parallel BC$. If $ABCD$ is a rectangle, then $f(ABCD) \leq 8$. If $ABCD$ is not a rectangle but a a parallelogram, then it is easy to see that $f(ABCD) \leq 4$. Now we may assume that $ABCD$ is a trapezoid with $|AD| = a < |BC|$ and an acute corner B. Let E be the orthogonal projection of A on the line $\overline{BC}$ and suppose that $|AE| = h$.

Case 1. $\angle C < \frac{\pi}{2}$.

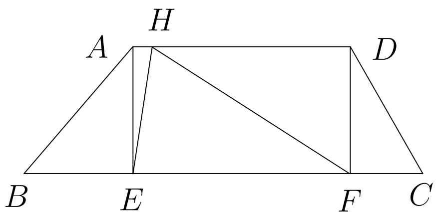


Figure 2.1 $\angle C < \frac{\pi}{2}$

Let F be the orthogonal projection of D on the line $\overline{BC}$. Clearly, $\{E, F\} \subseteq$ relintBC. Now let $H \in$ relintAD be a point closed to A such that the triangle HEF is acute. Then segments AE, EH, HF and FD divide $ABCD$ into 5 non-obtuse triangles, as shown in Figure 2.1. Now we slightly slide E in direction $\overrightarrow{BC}$ and F in direction $\overrightarrow{CB}$ such that all the triangles become acute.

Case 2. $\angle C = \frac{\pi}{2}$.

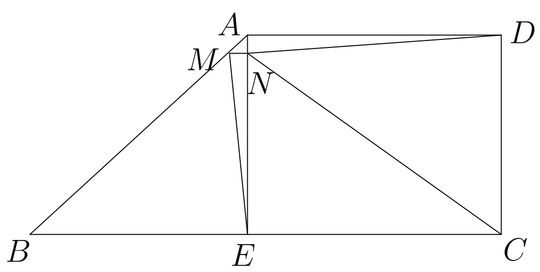


Figure 2.2 $\angle C = \frac{\pi}{2}$

Then $E \in \mathrm{relint}BC$. For any point $N \in AE$, let M be the point on AB such that $MN \perp AE$. Now we choose N close to A enough such that N lies out the circle with diameter CD and M lies out the circle with diameter BE. Then the trapezoid $ABCD$ admits a non-obtuse triangulation with size 6, as shown in Figure 2.2. Now we slightly slide N in direction $\overrightarrow{AD}$ and then we obtain an acute triangulation of $ABCD$ with size 6.

Case 3. $\angle C > \frac{\pi}{2}$.

Subcase 3.1. $E \in \mathrm{relint}BC$.

Let F be the orthogonal projection of C on the line $\overline{AD}$; then $C \in \mathrm{relint}AD$ since $E \in \mathrm{relint}BC$. We slightly slide E in direction $\overrightarrow{BE}$ and F in direction $\overrightarrow{DF}$, and obtain an acute triangulation of $ABCD$ with size 4.

Subcase 3.2. $E = C$.

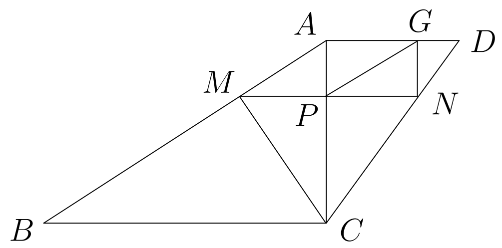


Figure 2.3 $\angle C > \frac{\pi}{2}$, $E = C$

Let M be the orthogonal projection of C on side AB. Let $N \in CD$ such that $MN \parallel BC$ and denote $MN \cap AC$ by P. Let G be the orthogonal projection of N on the line $\overline{AD}$. Then the line-segments PA, PM, PC,PN, PG, MC and NG divide the trapezoid $ABCD$ into 7 non-obtuse triangles, as shown in Figure 2.3. Let H be the midpoint of CN. Now we slightly slide P towards the side AD in the direction perpendicular to PH such that PMC, PCN, PNG and APG become acute, and $\angle APM + \angle MPC$ becomes less than π. Finally, we can slide M in direction $\overrightarrow{BA}$ and G in direction $\overrightarrow{DA}$ such that all the triangles become acute.

Subcase 3.3. $E \cap BC = \emptyset$.

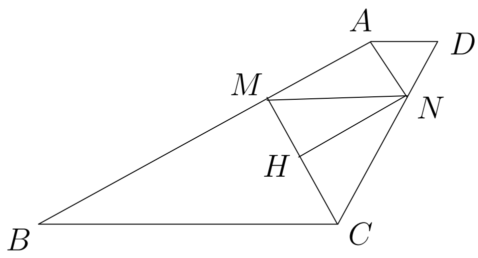


Figure 2.4 $\angle C > \frac{\pi}{2}$, $E \cap BC = \emptyset$

Let M be the orthogonal projection of C on the side AB. Let $N \in CD$ with $AN \parallel MC$. Then the line-segments AN, MN and CM divide $ABCD$ into 4 triangles, as shown in Figure 2.4. Firstly, we slightly slide M in direction $\overrightarrow{BM}$ and N in direction $\overrightarrow{DN}$ such that ANM and BMD become acute. Let H be a point on MC such that $\angle NHC = \frac{\pi}{2}$. Clearly $H \in \text{relint}CM$. By Proposition 2.2.2, there is an acute triangulation of acute triangle BMC with size 4 such that H is the only side vertex lying on MC. After slightly sliding H in direction $\overrightarrow{NH}$, we obtain an acute triangulation of $ABCD$ with size 8.

$\square$

Chapter 3. Acute Triangulations of Pentagons

In this chapter we discuss the acute triangulations of pentagons. In §3.1, we present some preparatory propositions. In §3.2, we prove the main result of this chapter, saying that every pentagon can be triangulated into at most 54 acute triangles.

§3.1 Preliminaries

The following results obtained by Maehara [34] will be very useful.

Proposition 3.1.1. ([34]) *Let $ABCD$ be a convex quadrilateral. If $\angle B < \frac{\pi}{2}$ and $\angle D \geq \frac{\pi}{2}$, then there is an acute triangulation $\mathcal{T}$ of $ABCD$ of size at most* 9 *such that there is no side vertex in $CD \cup DA$, B has degree* 2*, and each other vertex of $\mathcal{T}$ has degree at least* 3.

Proposition 3.1.2. ([34]) *Every quadrilateral admits an acute triangulation of size not larger than* 10 *with at most two side vertices on each side. Furthermore, there are at most* 2 *vertices with degree* 2*, and they are not adjacent.*

The following results will also be used.

Proposition 3.1.3. *Let ABC be a triangle with $\angle B < \frac{\pi}{2}$ and let M, $N \in$* relintAC. *Then ABC admits a non-obtuse triangulation of size at most* 11*, with M, N as the only side vertices on AC, so that the angles at all vertices different from M and N are acute.*

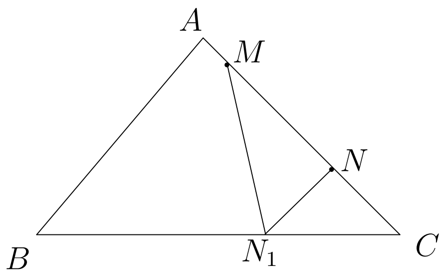


Figure 3.1 A non-obtuse triangulation of ABC with two points in relintAC

Proof. Consider $N_1 \in AB \cup BC$ with $N_1 N \perp AC$. We may assume without loss of generality that $\angle C < \frac{\pi}{2}$, $M \in \text{relint}AN$ and $N_1 \in \text{relint}BC$, as shown in Figure 3.1. By Proposition 3.1.1, the quadrilateral ABN_1M can be triangulated into at most 9 acute triangles with no new vertex introduced on $AM \cup MN_1$. Hence ABC admits a non-obtuse triangulation of size at most 11, with M, N as the only side vertices on AC. $\square$

A point $P \in \Gamma$ and an edge XY of Γ are said to be *facing* each other *in* Γ, if the points P, X, Y are the vertices of a non-degenerate triangle contained in Γ and $\angle PXY$, $\angle PYX$ are both not greater than $\frac{\pi}{2}$. A point $P \in \text{int}\Gamma$ is called a *pivot* of Γ if all edges of Γ are facing P in Γ.

The following result refines Proposition 1 in [35]; the similar proof is omitted.

Proposition 3.1.4. *If a convex polygon Γ has a pivot $P \in \text{int}\Gamma$, then it admits an acute triangulation in which the vertices newly introduced on the edges facing P are the orthogonal projections of P. Furthermore, if Γ has n vertices, m non-obtuse angles and r edges such that the orthogonal projection of P on each of them is one of its endpoints, then the number of triangles in this acute triangulation is at most $4n + 2m - r$.*

§3.2 Acute Triangulations of Pentagons

Theorem 3.2.1. *Every pentagon can be triangulated into at most* 54 *acute triangles.*

In order to prove Theorem 3.2.1, we first present some lemmas.

Lemma 3.2.2. *Every pentagon with at least one acute angle can be triangulated into at most* 21 *acute triangles. Furthermore, there are at most* 2 *vertex with degree* 2.

Proof. Let $\Gamma = ABCDE$. We may assume without loss of generality that $\angle B$ is acute. By Proposition 3.1.2, $ACDE$ admits an acute triangulation $\mathcal{T}$ with card$\mathcal{T} \leq 10$ such that there are at most 2 side vertices on AC.

Case 1. There is no side vertex on AC.

Then we assume that ACM is the acute triangle in $\mathcal{T}$ which contains AC. Let H be the orthogonal projection of M on AC. By Proposition 2.2.2, ABC can be triangulated into at most 7 acute triangles with H as the only side vertex on AC. Then we can slightly slide H in direction $\overrightarrow{MH}$ such that both triangles MAH and MCH become acute, and obtain an acute triangulation of Γ whose size is at most 18.

Case 2. There is precisely one side vertex on AC.

Then, by Proposition 2.2.2, Γ can be triangulated into at most 17 acute triangles.

Case 3. There are exactly two side vertices on AC.

We may assume that M, N are the two side vertices on AC. Use Proposition 3.1.3 to triangulate Γ into at most 21 non-obtuse triangles. Finally we can slightly slide M, N away from ABC in direction perpendicular to AC such that all the triangles become acute.

Also, by Proposition 3.1.2, it is easy to see that there are at most 2 vertex with degree 2. $\square$

Lemma 3.2.3. *Let ABE be a triangle with $AH \perp BE$ ($H \in$relintBE). Then for any two points $S \in$relintBH and $T \in$relintHE, ABE can be triangulated into at most 22 non-obtuse triangles such that the only side vertices on BE are S, H and T, and the angles at all vertices different from S, H and T are acute.*

Proof. Consider $S' \in$ relintAB, $T' \in$ relintAE with $S'S \perp BE$, $T'T \perp BE$.

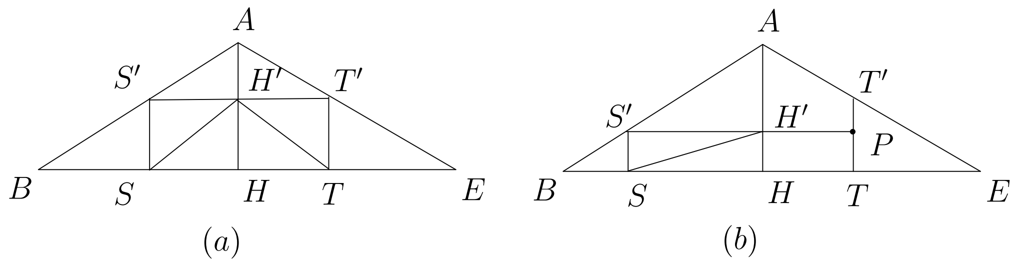


Figure 3.2 Three side vertices on BE

Case 1. $S'T' \parallel BE$.

Let $H' = S'T' \cap AH$. Then ABE can be triangulated into 8 right triangles as shown in Figure 3.2(a). Now we can slightly slide H' in direction $\overrightarrow{AH'}$ such that $AS'H'$, $S'SH'$, $AH'T'$, $H'TT'$ become acute.

Case 2. $S'T' \nparallel BE$.

We may assume without loss of generality that $|S'S| < |T'T|$. Let $S'P \parallel BE$, as shown in Figure 3.2(b), then P is a pivot of AHE. By Proposition 3.1.4, AHE can be triangulated into at most 18 acute triangles and therefore

ABE can be triangulated into 22 non-obtuse triangles such that the only side vertices on BE are S, H and T. Finally we can slightly slide H' in direction $\overrightarrow{AH'}$ such that only $S'BS$ and $H'SH$ are right triangles. $\square$

Lemma 3.2.4. *Let ABE be a triangle with $AH \perp BE$ ($H \in$relintBE). Then for any three points $S_1, S_2 \in$relintBH and $T \in$relintHE, ABE can be triangulated into at most 42 non-obtuse triangles such that the only side vertices on BE are S_1, S_2, H and T, and the angles at all vertices different from S_1, H and T are acute.*

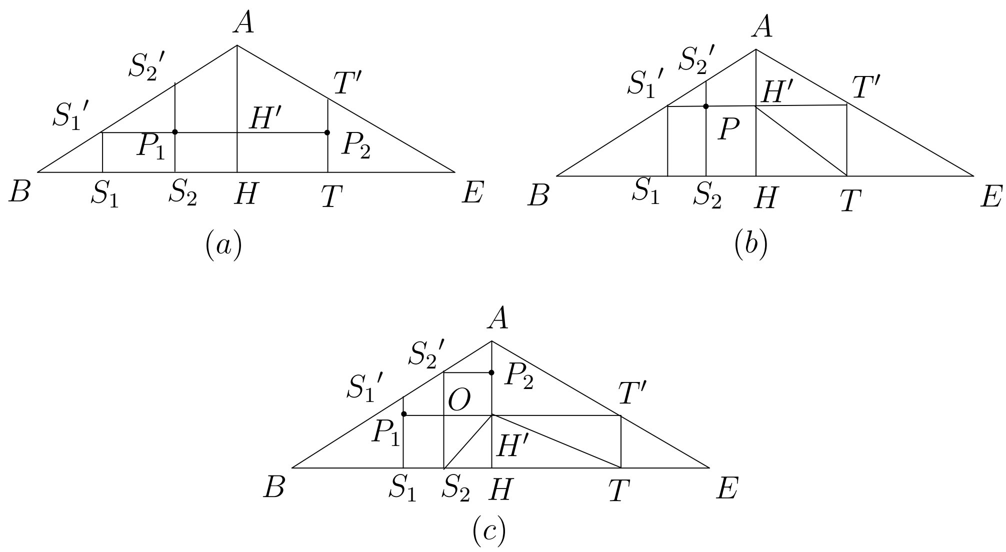


Figure 3.3 Four side vertices on BE

Proof. Consider ${S_1}', {S_2}' \in$ relintAB, $T' \in$ relintAE with ${S_1}'S_1 \perp BE$, ${S_2}'S_2 \perp BE$, $T'T \perp BE$.

Case 1. $|{S_1}'S_1| < |T'T|$.

Let ${S_1}'P_2 \parallel BE$, as shown in Figure 3.3 (a). Then P_1 (resp. P_2) is a pivot of $A{S_1}'S_1H$ (resp. AHE). By Proposition 3.1.4, $A{S_1}'S_1H$ (resp. AHE) can be triangulated into at most 21 (resp. 18) acute triangles and therefore ABE

admits a non-obtuse triangulation with size at most 40, where only $S_1{}'BS_1$ is a right triangle.

Case 2. $|S_1S_1{}'| = |TT'|$.

See Figure 3.3 (b). P is a pivot of $AS_1{}'S_1H$ and therefore ABE can be triangulated into at most 26 non-obtuse triangles. Now we can slightly slide H' in direction $\overrightarrow{AH'}$ such that only $S_1{}'BS_1$, $H'HT$ and $T'TE$ are right triangles.

Case 3. $|S_1S_1{}'| > |TT'|$.

Let $P_1T' \parallel BE$, $S_2{}'P_2 \perp AH$, as shown in Figure 3.3 (c), then P_1 (resp. P_2) is a pivot of $S_2{}'BS_2$ (resp. $AS_2{}'OT'$) and therefore ABE admits a non-obtuse triangulation with size at most 42. Finally we can slightly slide H' in direction $\overrightarrow{AH'}$ such that only $H'S_2H$, $H'HT$ and $T'TE$ are right triangles. $\square$

Lemma 3.2.5. *Every pentagon without any acute angle can be triangulated into at most* 54 *acute triangles.*

Proof. If the pentagon Γ has no acute angle, then it must be convex. Let $\Gamma = ABCDE$ be such a convex pentagon; we may assume without loss of generality that BE is the longest diagonal, which implies that $\angle EBC < \frac{\pi}{2}$, $\angle BED < \frac{\pi}{2}$. Let $AH \perp BE$ with $H \in BE$.

Case 1. Both $\angle BCH$ and $\angle EDH$ are less than $\frac{\pi}{2}$.

We may assume without loss of generality that $\angle EHD < \frac{\pi}{2}$. $\angle BCH < \frac{\pi}{2}$ implies that $BCDH$ can be triangulated into at most 10 acute triangles such that there is only one side vertex on BH. If there is no side vertex on DH, then let $DT \perp HE$, as shown in Figure 3.4 (a). We use Lemma 3.2.3 to triangulate Γ into at most 34 non-obtuse triangles. If there is a vertex on DH, then by Proposition 2.2.2 HDE can be triangulated into 4 acute triangles such that there is exactly one new vertex introduced on HE. Therefore we can trian-

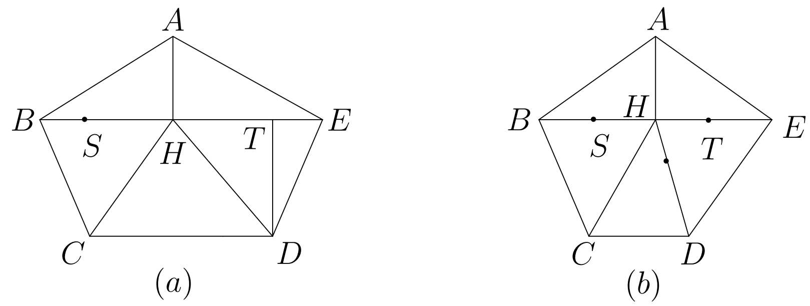


Figure 3.4 Both $\angle BCH$ and $\angle EDH$ are less than $\frac{\pi}{2}$

gulate Γ into at most 36 non-obtuse triangles. Finally, in both triangulations we slightly slide H in direction $\overrightarrow{AH}$ at first and then slightly slide S, T in direction $\overrightarrow{BS}$, $\overrightarrow{ET}$ respectively, and obtain the desired acute triangulations.

Case 2. Both $\angle BCH$ and $\angle EDH$ are not less than $\frac{\pi}{2}$.

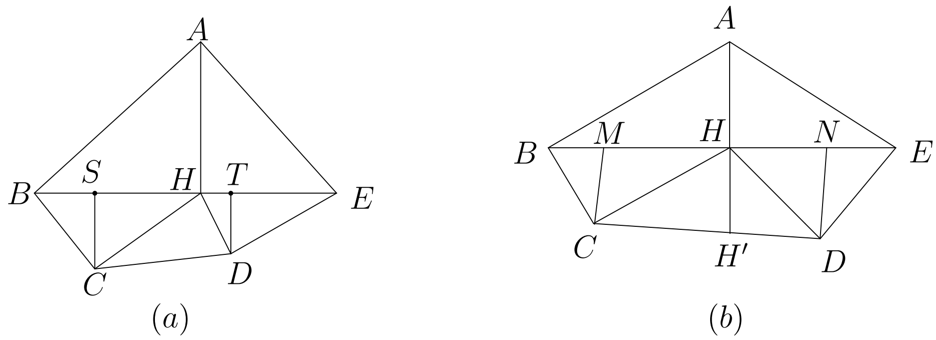


Figure 3.5 Both $\angle BCH$ and $\angle EDH$ are not less than $\frac{\pi}{2}$

If $\angle CHD < \frac{\pi}{2}$, then CHD is an acute triangle. Let $CS \perp BH$, $DT \perp HE$, as shown in Figure 3.5 (a). We use Lemma 3.2.3 to triangulate Γ into at most 27 non-obtuse triangles, which can be converted into acute triangles by the similar sliding used in Case 1.

If $\angle CHD \geq \frac{\pi}{2}$, then the extending line of AH must intersect the relative interior of CD at a point H', as shown in Figure 3.5 (b). Since $\angle HCH' < \frac{\pi}{2}$ and $\angle BCH' > \frac{\pi}{2}$, there is a point $M \in \text{relint}BH$ such that $\angle MCH' = \frac{\pi}{2}$.

Similarly, there is a point $N \in \text{relint} HE$ such that $\angle NDH' = \frac{\pi}{2}$. Thus M is a pivot of $ABCH'$ and N is a pivot of $AH'DE$. By Proposition 3.1.4, $ABCDE$ can be triangulated into at most 38 acute triangles.

Case 3. One of $\angle BCH$ and $\angle EDH$ is less than $\frac{\pi}{2}$ while the other is not.

We may assume without loss of generality that $\angle EDH < \frac{\pi}{2}$, $\angle BCH \geq \frac{\pi}{2}$.

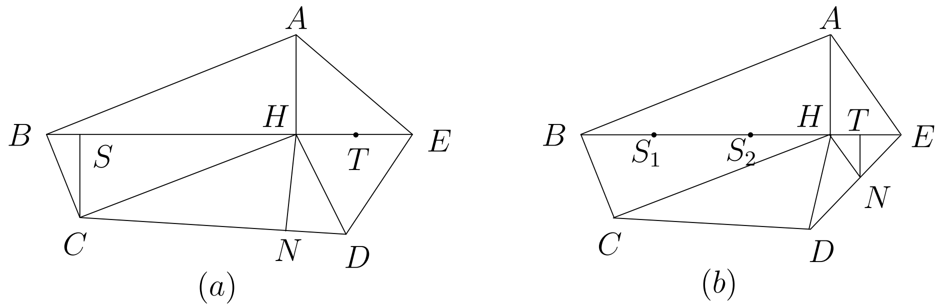


Figure 3.6 $\angle EDH < \frac{\pi}{2}$, $\angle BCH \geq \frac{\pi}{2}$.

If $\angle HDC$ is acute, then let $CS \perp BH$, $HN \perp CD$, as shown in Figure 3.6 (a). By Proposition 3.1.1 we know that $HNDE$ can be triangulated into at most 9 acute triangles such that there is exactly one new vertex introduced on HE. Now we slightly slide N in direction $\overrightarrow{CD}$ such that HCN becomes acute. We use Lemma 3.2.3 to triangulate Γ into at most 34 non-obtuse triangles, and all of them can be converted into acute by properly sliding of H, S and T.

If $\angle HDC \geq \frac{\pi}{2}$, then by Proposition 3.1.1, the quadrilateral $BCDH$ can be triangulated into at most 9 acute triangles such that there is no new vertex introduced on DH while there are exactly two new vertices introduced on BH, as shown in Figure 3.6 (b). Let $HN \perp DE$ and slightly slide N in direction $\overrightarrow{DE}$ such that HDN becomes acute. Let $NT \perp HE$. We use Lemma 3.2.4 to triangulate Γ into at most 54 non-obtuse triangles. Now we slightly slide H in direction $\overrightarrow{AH}$ at first and then slightly slide S_1, T in direction $\overrightarrow{BS_1}$, $\overrightarrow{ET}$ respectively, and obtain an acute triangulation of Γ. $\square$

Proof of Theorem 3.2.1.

Combine Lemma 3.2.2 and Lemma 3.2.5. $\square$

Chapter 4. Acute Triangulations of Polygons

§4.1 Introduction

Since the wide applications in computer graphics, physical simulation, and geographical information systems, several heuristic methods have been developed to compute non-obtuse triangulations of polygons [4, 41]. In 1988, Baker, Grosse, and Rafferty [3] gave the first provably correct algorithm. In 1992, Melissaratos and Souvaine presented another algorithm of this type. In the same year, Bern and Eppstein [7] devised a nonobtuse triangulation algorithm using $O(n^2)$ triangles, where n is the number of vertices of the input domain. This result demonstrates a fundamental complexity separation between bounding large angles and bounding small angles. In the same year, Bern, Dobkin and Eppstein [6] improved this bound to $O(n^{1.85})$ for convex polygons. In 1995, Bern, Mitchell and Ruppert [10] gave an algorithm for triangulating n-gons into $O(n)$ non-obtuse triangles. Later on, in 2002, Maehara [35] proved that every n-gon can be triangulated into at most $2 \cdot 6^5 N$ acute triangles, where N denotes the number of triangles in the existing non-obtuse triangulation.

In this chapter [48] we discuss the acute triangulations of any polygons. In Section §4.2, we present some preparatory propositions. In Section §4.3, we describe another way to acutely triangulate a polygon based on the existence of a non-obtuse triangulation. We obtain the upper bound $22N$ for the number of triangles in the acute triangulation. In Section §4.4, we find a concrete upper bound for the size of a non-obtuse triangulation of an n-gon, obtaining

$N \leq 106n - 216$ on the basis of the method used in [10]. This implies that every n-gon can be triangulated into at most $22 \cdot (106n - 216)$ acute triangles.

§4.2 Preliminaries

We shall make use of the following special triangulation of the rectangle.

Proposition 4.2.1. *Every rectangle can be triangulated into* 14 *acute triangles such that the side-vertices are precisely the* 4 *mid-points of the sides.*

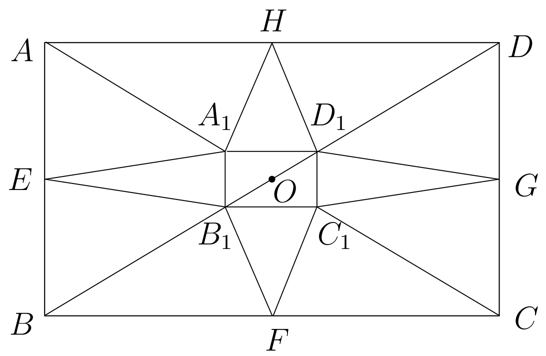


Figure 4.1 An acute triangulation of the rectangle

Proof. Let $ABCD$ be a rectangle with center O and let E, F, G, H be the midpoints of AB, BC, CD, DA respectively. Let B_1 be a point lying on OB which is very close to O, and let A_1, C_1, D_1 be the symmetric points of B_1 respecting to OE, OF, O respectively. If B_1 is close to O enough, then all the triangles except for $A_1B_1D_1$ and $B_1C_1D_1$ in Figure 4.1 are acute. Now we slightly slide A_1 in direction $\overrightarrow{OA}$ and C_1 in direction $\overrightarrow{OC}$ such that $A_1B_1D_1$ and $B_1C_1D_1$ become acute. Thus we obtain an acute triangulation of $ABCD$

with size 14 such that there is precisely one new vertex introduced on each side, and the new vertex is exactly the midpoint of the side. □

For the sake of convenience, we call this acute triangulation (described in Proposition 4.2.1) a *basic triangulation* of the rectangle.

Let $T = ABC$ be a triangle and M, N, P be the midpoints of AB, BC, CA respectively. Then MN, NP, MP divide ABC into 4 congruent triangles which are similar to T. We call such a triangulation an *elementary subdivision* of T.

§4.3 Acute Triangulations of Polygons

In a triangulation, two triangles sharing a common edge will be called *adjacent*. Let Γ be a polygon and $\mathcal{T}$ a non-obtuse triangulation of Γ. We describe now a new way of transforming $\mathcal{T}$ into an acute triangulation of Γ.

Step 1. Subdivide $\mathcal{T}$ to $\mathcal{T}_1$.

Rule 1.1: Any acute triangle in $\mathcal{T}$ will be divided into 4 acute triangles by an elementary subdivision.

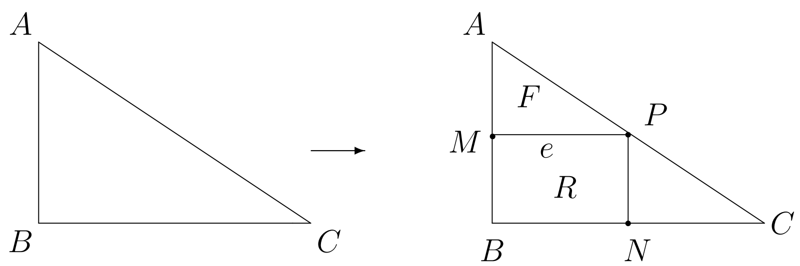


Figure 4.2 The division of a right triangle

Rule 1.2: Any right triangle in $\mathcal{T}$ will be divided into a rectangle and two right triangles as shown in Figure 4.2, where the point on each side is its midpoint.

Then we obtain a face-to-face tiling $\mathcal{T}_1$ of Γ which consists of acute triangles, rectangles and right triangles. Let $\mathcal{S}_1$, $\mathcal{R}_1$ and $\mathcal{F}_1$ denote the family of all acute triangles, all rectangles and all right triangles in $\mathcal{T}_1$. Then $\mathcal{T}_1 = \mathcal{S}_1 \cup \mathcal{R}_1 \cup \mathcal{F}_1$. Each triangle $F \in \mathcal{F}_1$ is obtained according to the Rule 1.2 (see Figure 4.2) and has a rectangle R as neighbor. This rectangle R will be called the *basic rectangle* of F and the common edge of F and R the *basic edge* of F. Thus any $F \in \mathcal{F}_1$ has one and only one basic rectangle and basic edge.

Step 2. Subdivide $\mathcal{T}_1$ into $\mathcal{T}_2$.

Rule 2.1: Apply an elementary subdivision to every element in $\mathcal{S}_1 \cup \mathcal{F}_1$.

Rule 2.2: Apply a basic triangulation to every rectangle in $\mathcal{R}_1$.

Then we obtain a non-obtuse triangulation $\mathcal{T}_2$ of Γ. Next we'll prove that $\mathcal{T}_2$ can be converted into an acute triangulation of Γ.

Let $\mathcal{S}_2$, $\mathcal{R}_2$, $\mathcal{F}_2$ denote all the triangles in $\mathcal{T}_2$ which are obtained from $\mathcal{S}_1$, $\mathcal{R}_1$, $\mathcal{F}_1$ respectively. Every triangle in $\mathcal{S}_2 \cup \mathcal{R}_2$ is acute. Hence we only need to consider the triangles in $\mathcal{F}_2$. Notice that the right triangles in $\mathcal{F}_2$ are obtained from $\mathcal{F}_1$ by elementary subdivision, so we can classify them according to their corresponding right triangles in $\mathcal{F}_1$.

For any acute triangles $S \in \mathcal{S}_1$, let $\phi(S)$ denote the family of four acute triangles obtained from the elementary subdivision of S. For any rectangle $R \in \mathcal{R}_1$, let $\phi(R)$ denote the family of 14 acute triangles obtained from the basic triangulation of R. For any right triangles $F \in \mathcal{F}_1$, let $\phi(F)$ denote the family of four right triangles obtained from the elementary subdivision of F.

We call the three triangles in $\phi(F)$ which have non-empty intersection with the basic edge e of F the *adjacent triangles* with respect to e, and the other one which is disjoint from e the *opposite triangle* with respect to e. Moreover, if $\mathcal{S}_1 = \{S_1, S_2, \cdots, S_t\}$, $\mathcal{F}_1 = \{F_1, F_2, \cdots, F_n\}$ and $\mathcal{R}_1 = \{R_1, R_2, \cdots, R_m\}$, then $\mathcal{S}_2 = \phi(S_1) \cup \phi(S_2) \cup \cdots \cup \phi(S_t)$, $\mathcal{F}_2 = \phi(F_1) \cup \phi(F_2) \cup \cdots \cup \phi(F_n)$ and $\mathcal{R}_2 = \phi(R_1) \cup \phi(R_2) \cup \cdots \cup \phi(R_m)$.

Lemma 4.3.1. *For any $F \in \mathcal{F}_1$ with basic rectangle $R \in \mathcal{R}_1$ and basic edge e, the triangulation can be perturbed so that all triangles in $\phi(F) \cup \phi(R)$ except the opposite triangle with respect to e (in $\phi(F)$) become or remain acute.*

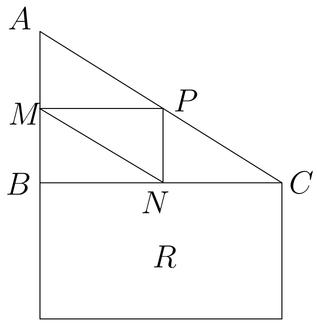


Figure 4.3 The basic sliding of $\phi(F)$

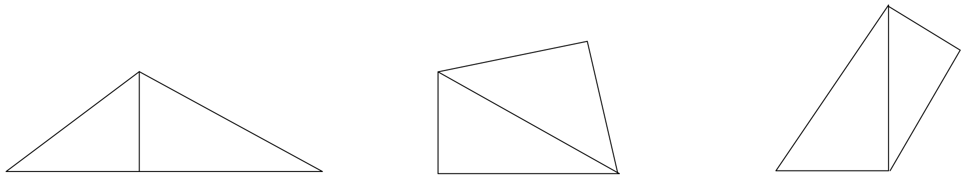
Proof. Let $F = ABC$ with basic edge BC, and let N be the midpoint of BC, as shown in Figure 4.3. We can slightly slide N in direction $\overrightarrow{NB}$ such that both MPN and PNC become acute triangles while all triangles in $\phi(R)$ remain acute. Then we slightly slide N in direction $\overrightarrow{BA}$ such that MBN becomes acute while MPN, PNC and all triangles in $\phi(R)$ remain acute. □

For each $F \in \mathcal{F}_1$, we call the sliding described in the proof of Lemma 4.3.1 the *basic sliding* of $\phi(F)$.

Let $\mathcal{F}_1 = \{F_1, F_2, \cdots, F_n\}$, and e_i denote the basic edge of F_i ($i = 1, 2, \cdots, n$). Next we convert the triangulation $\mathcal{T}_2$ to $\mathcal{T}_3$ by the following step.

Step 3. Apply the basic sliding to each $\phi(F_i)$, where $i = 1, 2, \cdots, n$.

After Step 3, we know that in each $\phi(F_i)$ ($i = 1, 2, \cdots, n$), only the opposite triangle with respect to e_i is not changed into an acute triangle. We denote by $\phi_3(F_i)$ the set of four triangles in $\mathcal{T}_3$ resulting from $\phi(F_i)$. Then, there are exactly n right triangles in $\mathcal{T}_3$, one in each of $\phi_3(F_i)$ ($i = 1, 2, \cdots, n$).



(a) Type I (b) Type $I\!I$ (c) Type $I\!I\!I$

Figure 4.4 Three kinds of adjacent triangles in $\mathcal{F}_1$

If $F \in \mathcal{F}_1$ has no adjacent right triangle, then we call it an *isolated triangle*. Let F_i and F_j be two adjacent triangles in $\mathcal{F}_1$, if they have a common leg (hypotenuse), we call them of *Type I* (or of *Type II*); Otherwise, we call them of *Type III*, see Figure 4.4.

Lemma 4.3.2. *For any two adjacent triangles F_i and F_j in $\mathcal{F}_1$, we can transform all the triangles in $\phi_3(F_i) \cup \phi_3(F_j)$ into acute triangles, leaving unchanged all other triangles in $\mathcal{T}_3 - (\phi_3(F_i) \cup \phi_3(F_j))$.*

Proof. There are three cases to consider.

Case 1. F_i and F_j are of Type I.

From the given condition we know that e_i and e_j must be collinear. Suppose that $F_i = ABC$ and $F_j = ACD$, as shown in Figure 4.5. Then $e_i = BC$,

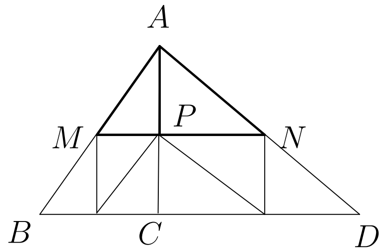


Figure 4.5 F_i and F_j are of Type I

$e_j = CD$. Thus AMP and ANP are the two remaining right triangles in $\phi_3(F_i) \cup \phi_3(F_j)$. Now we can slightly slide P in direction $\overrightarrow{AC}$ such that both AMP and ANP become acute while all other triangles in $\phi_3(F_i) \cup \phi_3(F_j)$ remain acute.

Case 2. F_i and F_j are of Type $I\!I$.

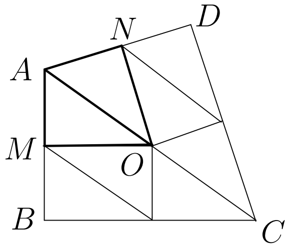


Figure 4.6 F_i and F_j are of Type $I\!I$

Without loss of generality, we may assume that $F_i = ABC$ with basic edge BC and $F_j = ACD$. Then $e_j = CD$. Hence AMO and ANO are the two remaining right triangles in $\phi_3(F_i) \cup \phi_3(F_j)$, as shown in Figure 4.6. Now we can slightly slide O in direction $\overrightarrow{CA}$ such that all the triangles in $\phi_3(F_i) \cup \phi_3(F_j)$ are acute.

Case 3. F_i and F_j are of Type $I\!I\!I$.

Without loss of generality, we may assume that the common edge is a leg of F_i and the hypotenuse of F_j.

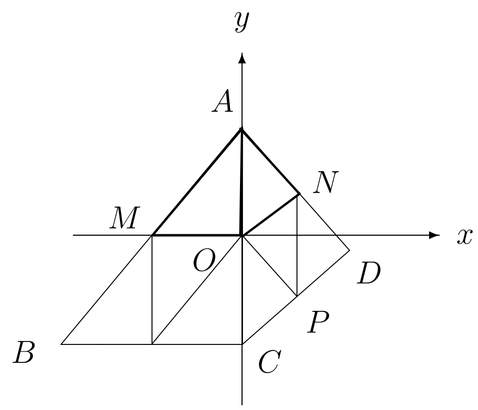


Figure 4.7 F_i and F_j are of Type *III*

Let $F_i = ABC$ and $F_j = ACD$, then $e_i = BC$, $e_j = CD$. Thus AMO and ANO are the two remaining right triangles in $\phi_3(F_i) \cup \phi_3(F_j)$, as shown in Figure 4.7 . We establish an x-y coordinate system with O as origin, MO as $x-$axis and CA as $y-$axis (also see Figure 4.7). Let k_{ON} denote the slope of the line $\overline{ON}$. Then $k_{ON} \neq 0$. Let **O** be the circumscribed circle of AMO and l be the tangent line of **O** at point O, then the slope k_l of l is not zero.

If $k_{ON} \leq k_l$, we can slightly slide O in direction $\overrightarrow{ON}$ at first such that AMO becomes acute while ANO remains right and the rest of triangles in $\phi_3(F_i) \cup \phi_3(F_j)$ remain acute. Then we slightly slide O in direction $\overrightarrow{DA}$ such that all the triangles in $\phi_3(F_i) \cup \phi_3(F_j)$ are acute.

If $k_{ON} > k_l$, similarly we can slightly slide O in direction $\overrightarrow{NO}$ at first and then slightly slide O in direction $\overrightarrow{DA}$ such that all the triangles in $\phi_3(F_i) \cup \phi_3(F_j)$ are acute.

Finally, it is easy to check that all the slidings mentioned above leave unchanged all other triangles in $\mathcal{T}_3 - (\phi_3(F_i) \cup \phi_3(F_j))$. The proof is complete. $\square$

For any two adjacent triangles F_i and F_j in $\mathcal{F}_1$, we call the sliding described in the proof of Lemma 4.3.2 the *basic pair sliding* of F_i and F_j.

Step 4. Apply the basic pair sliding in $\mathcal{T}_3$ as follows, and denote the obtained triangulation by $\mathcal{T}_4$:

Step 4.1. Apply the basic pair sliding to all type I adjacent triangles in $\mathcal{F}_1$, and denote all the triangles involved in this step by $\mathcal{P}_1$;

Step 4.2. Apply the basic pair sliding to all type II adjacent triangles in $\mathcal{F}_1 - \mathcal{P}_1$, and denote all the triangles involved in this step by $\mathcal{P}_2$;

Step 4.3. If there are two triangles in $\mathcal{F}_1 - (\mathcal{P}_1 \cup \mathcal{P}_2)$ which are adjacent triangles of type $I\!I\!I$, then we apply the basic pair sliding to them. Repeating this kind of work until there is no type $I\!I\!I$ adjacent triangles anymore. Denote all the triangles involved in this step by $\mathcal{P}_3$.

Let $\mathcal{P}_0 = \mathcal{F}_1 - (\mathcal{P}_1 \cup \mathcal{P}_2 \cup \mathcal{P}_3)$. If $\mathcal{P}_0 = \emptyset$, then $\mathcal{T}_4$ is an acute triangulation of Γ. If $\mathcal{P}_0 \neq \emptyset$, then any $F \in \mathcal{P}_0$ is either an isolated triangle in $\mathcal{F}_1$ or its adjacent triangle (of Type $I\!I\!I$) admits an acute triangulation. Let $F = ABC$ be a triangle in $\mathcal{P}_0$ with basic edge BC. If there is an $X \in \mathcal{T}_1$ which has the common side AB with F, then all the triangles in $\phi(X)$ are acute. As a result, we can slightly slide N in direction $\overrightarrow{AB}$ such that all the triangles in $\phi_3(F) \cup \phi_3(X)$ become acute (see Figure 4.8). If there is not such an X, then AB lies on a side of Γ and we can perform the similar sliding. Applying this kind of sliding to each element of $\mathcal{P}_0$, we transform $\mathcal{T}_4$ into an acute triangulation of Γ.

Combining the above discussion, we immediately have the following theorem.

Theorem 4.3.3. *Every polygon admits an acute triangulation.*

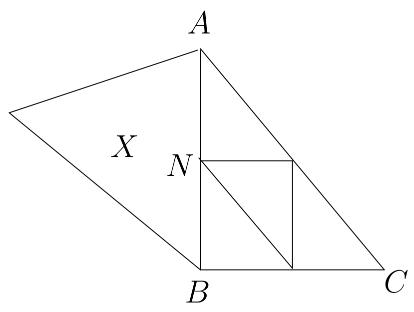


Figure 4.8 A triangle in $\mathcal{P}_0$

□

Furthermore, considering the number of triangles in the obtained acute triangulation, we have the following result.

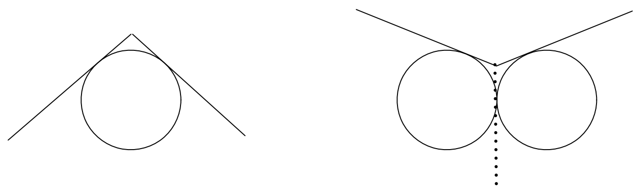
Theorem 4.3.4. *If a polygon can be triangulated into N non-obtuse triangles, then it can be triangulated into at most $22N$ acute triangles.*

Proof. Let $\mathcal{T}$ be a non-obtuse triangulation of a polygon and let $\nu_1(\nu_2)$ denote the number of all acute (right) triangles in $\mathcal{T}$; thus $N = \nu_1 + \nu_2$. Notice that card$\mathcal{S}_1 = 4\nu_1$, card$\mathcal{R}_1 = \nu_2$ and card$\mathcal{F}_1 = 2\nu_2$ in $\mathcal{T}_1$; we have card$\mathcal{T}_2 = 4(\text{card}\mathcal{S}_1 + \text{card}\mathcal{F}_1) + 14\ \text{card}\mathcal{R}_1 = 16\nu_1 + 22\nu_2 = 16(\nu_1 + \nu_2) + 6\nu_2 \leq 22N$. This implies card$\mathcal{T}_4$ =card$\mathcal{T}_3$ =card$\mathcal{T}_2 \leq 22N$. The proof is complete. □

§4.4 Non-obtuse Triangulations of Polygons

In [10] it was proved that every n-gon can be triangulated into $O(n)$ acute triangles. In this section we obtain a concrete upper bound for the size of a non-obtuse triangulation of an n-gon basing on work in [10].

Let Γ be an n-gon (without hole) with r concave corners. We start with the basic method, namely, disk packing presented in [10].



(a) (b)

Figure 4.9 Adding disks at corners

Firstly, we pack the disks at corners. At every convex vertex of Γ, we add a small disk tangent to both edges, as shown in Figure 4.9(a). At every concave corner of Γ, we add two disks of equal radii, tangent to the edges, and tangent to the angle bisector at the corner, as shown in Figure 4.9(b). We choose radii small enough such that the disks lie within Γ, and any two disks from two distinct vertices are disjoint. This step isolates a small 3- or 4-sided remainder region at each corner of Γ. The large remainder region is a simply connected curved polygon with $2n + r$ sides, where a *curved polygon* is a topological disc with a finite union of line-segments and arcs of circles as boundary (the circle may have various radii).

Let A be a simply connected curved m-gon. To subdivide A, we add a disk tangent to three sides, not all of which are consecutive (it is possible that it is tangent to more than three sides), as shown in Figure 4.10. Then we have the following result.

Lemma 4.4.1. ([10]) *It is possible to reduce the numbers of sides of each of the remainder regions to at most* 4*, by packing at most* $m - 4$ *non-overlapping disks into the arc-gon* A*.*

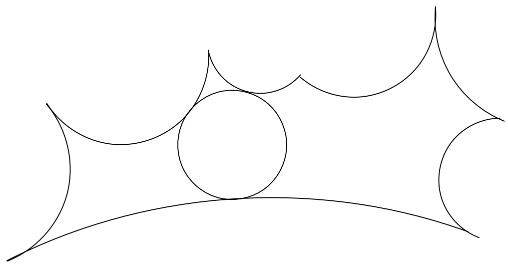


Figure 4.10 A disk tangent to three edges of an arc-gon

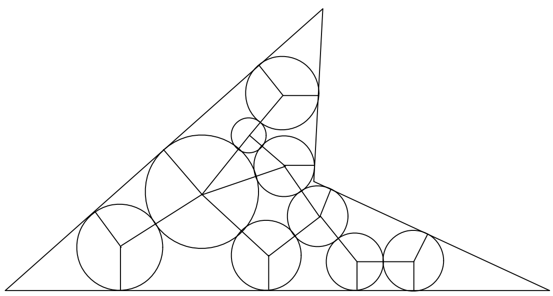


Figure 4.11 Decomposition of Γ into simple polygons with disjoint interiors

At this point, the polygonal region Γ has been partitioned into disks and remainder regions with three or four sides, either straight or circular arcs. It is clear that there are three kinds of such regions, namely, remainder regions with vertices of Γ (denoted by R_{P_i}, where P_i is a vertex of Γ $(i = 1, 2, \cdots, n)$), remainder regions with three sides (denoted by R_3) and remainder regions with four sides (denoted by R_4). Moreover, each circular arc of a remainder region R is naturally associated with a pie-shaped sector. We denote the union of R and its associated sectors by R^+. These *augmented* remainder regions define a decomposition of Γ into simple polygons with disjoint interiors, as shown in Figure 4.11.

Now we present several results in [10] concerning the triangulation of each R^+ obtained. New vertices introduced to triangulate Γ are called *additional points* (Steiner points in the terminology of [10]). We call an additional point in an augmented region R^+ *safe* if it lies either in the interior to R^+ or on the boundary of Γ.

Lemma 4.4.2. ([10]) *Let P_i $(i = 1, 2, \cdots, n)$ be a vertex of an n-gon Γ. If P_i is a convex (resp. concave) corner, then ${R^+}_{P_i}$ can be triangulated into* 2 *(resp.* 4*) right triangles such that there is no new vertex introduced in the triangulation.*

Lemma 4.4.3. ([10]) *Every ${R_3}^+$ (resp. ${R_4}^+$) can be triangulated into at most* 6 *(resp.* 28*) right triangles, adding only safe additional points.*

Combining Lemma 4.4.1 to Lemma 4.4.3 we know that for every $m-$arc-gon A, A^+ admits a non-obtuse triangulation (adding only safe additional points). Therefore Γ admits a non-obtuse triangulation. Let $\phi(m)$ denote the number of triangles in a non-obtuse triangulation of A^+ (where A is an $m-$arc-gon), then Lemma 4.4.3 implies that $\phi(3) \leq 6$ and $\phi(4) \leq 28$. Moreover, we have

Lemma 4.4.4. *For every $m \geq 4$, $\phi(m) \leq 34m - 108$.*

Proof. We prove this by induction on m. The base is $\phi(4) \leq 28$. Now we suppose that the conclusion holds for all $4 \leq k < m$.

Let A be an arc-gon with m sides, and let $\mathbf{O}$ be the first disk packed in A. We assume that λ ($\lambda \geq 3$) sides of A are tangent to $\mathbf{O}$. Then A is divided into one disk and λ arc-gons having $k_1, k_2, \cdots, k_\lambda$ sides, respectively, with $3 \leq k_1 \leq k_2 \leq \cdots \leq k_\lambda$. Thus $\phi(m) = \phi(k_1) + \phi(k_2) + \cdots + \phi(k_\lambda)$. Since the disk divides λ sides, and is itself divided in λ places, we have $k_1 + k_2 + \cdots + k_\lambda = m + 2\lambda$.

Case 1. $\lambda = 3$.

Then by the rules of packing disks we know that not all of these three sides are consecutive in A which implies that $3 \leq k_1 \leq k_2 \leq k_3 < m$.

If $k_1 = 3$, then $\phi(m) = \phi(3) + \phi(k_2) + \phi(k_3) \leq 6 + 34 \times (k_2 + k_3) - 2 \times 108 = 6 + 34 \times (m + 3) - 2 \times 108 = 34m - 108$.

If $k_1 \geq 4$, then $\phi(m) = \phi(k_1) + \phi(k_2) + \phi(k_3) \leq 34 \times (k_1 + k_2 + k_3) - 3 \times 108 = 34 \times (m + 6) - 3 \times 108 = 34m - 120 \leq 34m - 108$.

Case 2. $\lambda \geq 4$.

Then it is easy to see that $k_\lambda < m$. We may assume that among all of the k_i ($i = 1, \cdots, \lambda$), there are exactly λ_0 values which are equal to 3, where $0 \leq \lambda_0 \leq \lambda$. Then we have

$\phi(m) = \phi(k_1) + \phi(k_2) + \cdots + \phi(k_\lambda) \leq 6\lambda_0 + 34 \cdot (k_{\lambda_0+1} + \cdots + k_\lambda) - 108 \cdot (\lambda - \lambda_0) = 6\lambda_0 + 34 \cdot (m + 2\lambda - 3\lambda_0) - 108 \cdot (\lambda - \lambda_0) = 34m - 40\lambda + 12\lambda_0 \leq 34m - 28\lambda \leq 34m - 28 \times 4 = 34m - 112 \leq 34m - 108$.

The proof is complete. □

Theorem 4.4.5. *Every n-gon admits a non-obtuse triangulation whose size is at most* $106n - 216$.

Proof. Suppose that an n-gon Γ admits a non-obtuse triangulation (obtained by the method described above) with size N. Let r denote the number of concave corners of Γ. From the above discussion we know that $P = R^+{}_{P_1} \cup R^+{}_{P_2} \cup \cdots \cup R^+{}_{P_n} \cup R^+{}_A$, where A is an arc-gon with $2n+r$ sides. By Lemma 4.4.2 and Lemma 4.4.4 we know that $N = 2(n-r) + 4r + \phi(2n+r) \leq 2n + 2r + 34(2n+r) - 108 = 70n + 36r - 108$. Noticing that $r \leq n-3$, we have $N \leq 106n - 216$.

The proof is complete. $\square$

Combining Theorem 4.3.4 and Theorem 4.4.5, we immediately obtain the following corollary.

Corollary 4.4.6. *Every n-gon can be triangulated into at most* $22 \cdot (106n - 216)$ *acute triangles.*

Chapter 5. Acute Triangulations of Surfaces

§5.1 Introduction

In 2000 Hangan, Itoh and Zamfirescu [25] considered the following problem: does there exist a number N such that every compact convex surface in $\mathbb{R}^3$ admits an acute triangulation with at most N triangles? Of course, one should estimate N, if it exists. Indeed, among all the compact convex surfaces the polyhedral surfaces play a central role. In the same year, Hangan, Itoh and Zamfirescu started the investigation of acute triangulations of all Platonic surfaces, which are the surfaces of the five well-known Platonic solids. They [25] proved that the surface of the cube admits an acute (resp. non-obtuse) triangulation with 24 (resp. 4) triangles and no acute (resp. non-obtuse) triangulation with fewer triangles. In 2004 Itoh and Zamfirescu [26] proved that the surface of the regular icosahedron admits an acute (resp. non-obtuse) triangulation with 12 (resp. 8) triangles and no acute (resp. non-obtuse) triangulation with fewer triangles. They [27] also proved that the surface of the regular dodecahedral admits a non-obtuse triangulation with 10 triangles and no non-obtuse triangulation with fewer triangles, while it admits an acute triangulation with 14 triangles and no acute triangulation with less than 12 triangles.

At the same time, acute triangulations of some smooth convex sets were also considered, so for example, the sphere. The sphere itself can be acutely triangulated with at least 20 triangles, which can easily be deduced from Euler's formula together with the condition that valency is at least 5. In 2002 Itoh and Zamfirescu [28] investigated acute triangulations of triangles on the sphere, and proved that every proper triangle (a triangle with all angles less than π) admits an acute triangulation of size at most 10 and every non-proper triangle with all side lengths smaller than π admits an acute triangulation of size 18. They also pointed out that both estimates are best possible.

However, the case of arbitrary convex surfaces is much more difficult, even for a polyhedra with small number of vertices. So, for example, even the family of all tetrahedral surfaces is far from being easy to treat. We study here the acute triangulations of the doubly covered convex set, or simply the *double convex set*, which is a (degenerate convex) surface Γ_d homeomorphic to the sphere consisting of two planar isometric convex sets, Γ and Γ', with boundaries glued in the obvious way. For any point P in Γ, let P' denote the corresponding point in Γ'. Even this case is, in full generality, still too difficult. In 2004 C. Zamfirescu [52] considered acute triangulations of doubly covered triangles, and obtained the best estimate 12.

In this Chapter we continue this task and acutely triangulate several surfaces. In §5.2 we prove that every doubly covered quadrilateral admits an acute triangulation of size at most 20. In §5.3 we discuss acute triangulations of doubly covered pentagons [47] and obtain the upper bound 76. In §5.4 we discuss the acute triangulations of the double of an arbitrary symmetric smooth convex set. Then, we consider acute triangulations of several well-know surfaces. In §5.5 we prove that every flat torus admits an acute triangulation with size at most 16. In §5.6 we obtain the minimal size of acute triangulations of flat Möbius strips in several different cases.

§5.2 Acute Triangulations of Double Convex Quadrilaterals

In this section we consider the case when Γ_d is a double convex quadrilateral.

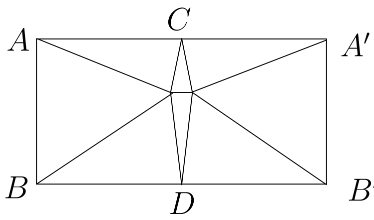


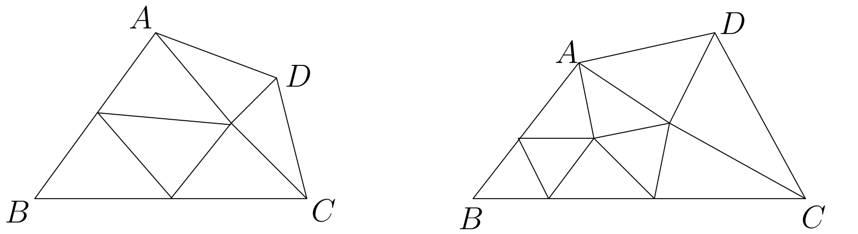
Figure 5.1 Γ is a rectangle

Theorem 5.2.1. *Every double convex quadrilateral admits an acute triangulation with size at most* 20.

Proof. If $\Gamma = ABCD$ is a rectangle, then Γ_d admits an acute triangulation with size 8, as shown in Figure 5.1, where the two "sides" of Γ_d are unfolded on a plane.

Now suppose that Γ is not a rectangle. Then it has at least one acute corner, say B.

If $\angle D \geq \frac{\pi}{2}$, then by Proposition 3.1.1 Γ admits an acute triangulation $\mathcal{T}$ with size at most 9 [34], in which the only vertex of degree less than 3 is B (Figure 5.2). Thus Γ_d can be divided into at most 18 acute triangles. Notice that this division is not a triangulation. Choose F, G such that both BF and FG are edges of $\mathcal{T}$ lying in bdΓ. We slightly slide F into the interior of Γ such that all the triangles in Γ having F as a vertex remain acute, and both BFF'



(a) Triangle ABC is acute (b) Triangle ABC is non-acute

Figure 5.2 Acute triangulations of a quadrilateral $ABCD$

and GFF' are acute as well. Thus Γ_d admits an acute triangulation of size at most 20.

If $\angle D < \frac{\pi}{2}$, we first prove that Γ can be triangulated into at most 8 acute triangles such that there are at most 2 vertices with degree 2.

If both triangles ABC and ACD are acute, then clearly Γ can be triangulated into 2 acute triangles.

If one and only one of them is acute, we may assume without loss of generality that the triangle ABC is acute and $\angle ACD \geq \frac{\pi}{2}$. Let E be the orthogonal projection of C on the side AD and F be the the orthogonal projection of E on the side AC. Clearly $E \in$relintAD and $F \in$relintAC. By Proposition 2.2.2, ABC can be triangulated into 4 acute triangles such that F is the only side vertex on AC. Now we slightly slide F in direction $\overrightarrow{EF}$ and E in direction $\overrightarrow{DE}$, and obtain an acute triangulation of Γ of size 7, in which only the vertices B and D have degree 2.

If both triangles ABC and ACD are non-acute, we may assume without loss of generality that $\angle ACB \geq \frac{\pi}{2}$, $\angle CAD \geq \frac{\pi}{2}$, and the lines including BA and CD intersect at some point closer to A than to B. Let M be the orthogonal projection of C on the side AB. Let $N \in CD$ with $AN \parallel MC$. Then the

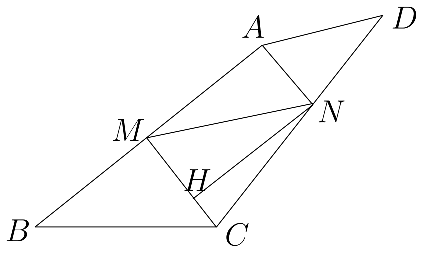


Figure 5.3 Both ABC and ACD are non-acute

line-segments AN, MN and CM divide $ABCD$ into 4 triangles, as shown in Figure 5.3. Firstly, we slightly slide M in direction $\overrightarrow{BM}$ and N in direction $\overrightarrow{DN}$ such that ANM and BMD become acute. Let H be a point on MC such that $\angle NHC = \frac{\pi}{2}$. Clearly $H \in \mathrm{relint}CM$. By Proposition 2.2.2, there is an acute triangulation of the acute triangle BMC with size 4 such that H is the only side vertex lying on MC. After slightly sliding H in direction $\overrightarrow{NH}$, we obtain an acute triangulation of $ABCD$ with size 8. Noticing that there are at most 2 vertices with degree 2 in each triangulation, we can conclude that Γ_d admits an acute triangulation with size at most 20. □

§5.3 Acute Triangulations of Double Convex Pentagons

In this section we consider case when Γ_d is a double convex pentagon.

Lemma 5.3.1. *If the convex pentagon* Γ *has at least one acute angle, then* Γ_d *can be triangulated into at most* 46 *acute triangles.*

Proof. By Lemma 3.2.2, Γ admits an acute triangulation $\mathcal{T}$ with size at most 21. Furthermore, there are at most 2 vertices in $\mathcal{T}$ with degree 2. Obviously Γ_d can be divided into at most 42 acute triangles (which may not form a proper triangulation). Now let A be a vertex with degree 2 in $\mathcal{T}$. This vertex belongs to two congruent triangles T, T', one on each face of Γ_d. Now suppose that $T = T' = \triangle EAF$, and $FG \subset \mathrm{bd}\Gamma$ is an edge of $\mathcal{T}$. Now we slightly slide F into the interior of Γ in direction perpendicular to AF such that all the triangles in Γ having F as a vertex remain acute, and both of AFF' and GFF' are acute as well. Recalling that there are at most 2 vertices in $\mathcal{T}$ with degree 2, we can conclude that Γ_d can be triangulated into at most 46 acute triangles. $\square$

Lemma 5.3.2. *Consider the side AB of Γ and $H \in \mathrm{int}\Gamma$ satisfying $\angle AHB > \frac{\pi}{2}$. Let $\mathcal{D}_{AB} = ABH \cup ABH'$. If $M \in \mathrm{relint}AH \cup \mathrm{relint}BH$, then $\mathcal{D}_{AB}$ admits a triangulation with precisely the points M, M' as side vertices and at most*

(*i*) 20 *non-obtuse triangles if ABH has two angles smaller than $\frac{\pi}{4}$;*

(*ii*) 30 *non-obtuse triangles otherwise.*

Proof. By unfolding $\mathcal{D}_{AB}$ in the plane, we obtain a quadrilateral $HAH'B$ with $AB \cap HH' = O$. We may assume without loss of generality that $M \in \mathrm{relint}AH$. Since $\angle AHB$ is obtuse, there is a point $U \in \mathrm{relint}\ AO$ such that $UH \perp HB$. Let l denote the line passing through M and perpendicular to AH.

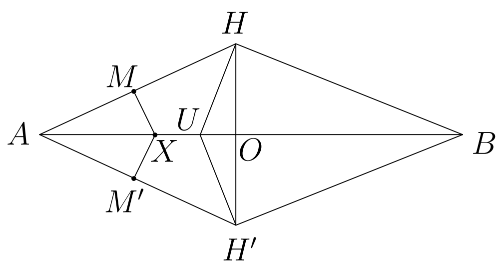


Figure 5.4 $l \cap (\mathrm{relint}AU \cup \{U\}) = \{X\}$

Case 1. $l \cap (\mathrm{relint}AU \cup \{U\}) = \{X\}$.

Then X is a pivot of $AH'H$ and therefore $AH'H$ can be triangulated into at most 18 acute triangles such that only one new vertex O is introduced on

HH'. Now we slightly slide O in direction $\overrightarrow{BO}$. So $\mathcal{D}_{AB}$ admits a triangulation with at most 20 acute triangles in which only M, M' are side vertices.

Case 2. $l \cap (\text{relint}AU \cup \{U\}) = \emptyset$.

We suppose that $l \cap \text{relint}HU = \{Y\}$.

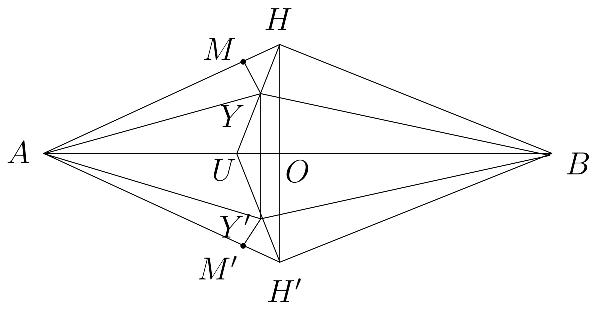


Figure 5.5 $l \cap (\text{relint}AU \cup \{U\}) = \emptyset$

(i) If both acute angles of ABH are less than $\frac{\pi}{4}$, then $\mathcal{D}_{AB}$ can be triangulated into 8 non-obtuse triangles AYM, $AY'M'$, HMY, $H'M'Y'$, HYB, $H'Y'B$, BYY' and AYY', as shown in Figure 5.5. Now we slightly slide Y in direction $\overrightarrow{MY}$ (and Y' in direction $\overrightarrow{M'Y'}$) such that only the four triangles adjacent to M or M' are right triangles.

(ii) Otherwise, it is easy to check that Y is a pivot of ABH. By Proposition 3.1.4, ABH can be triangulated into at most 15 acute triangles and therefore $\mathcal{D}_{AB}$ can be triangulated into at most 30 acute triangles such that M, M' are the only side vertices. □

Theorem 5.3.3. *Every double convex pentagon can be triangulated into at most* 76 *acute triangles.*

Proof. If Γ has acute angles, the conclusion follows from Lemma 5.3.1.

If Γ has no acute angle, then it has at most two angles which are greater than or equal to $\frac{3\pi}{4}$.

Case 1. Two angles of Γ are greater than or equal to $\frac{3\pi}{4}$.

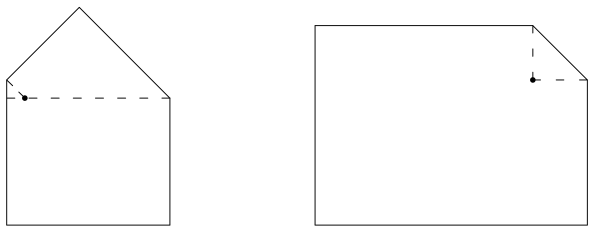


Figure 5.6 Two angles of Γ are greater than or equal to $\frac{3\pi}{4}$

Then Γ has two angles equal to $\frac{3\pi}{4}$ and three angles equal to $\frac{\pi}{2}$. So Γ has two possible non-isomorphic configurations as shown in Figure 5.6, and there is a pivot in Γ° for each of them. By Proposition 3.1.4 Γ can be triangulated into at most 26 acute triangles and therefore Γ_d can be triangulated into at most 52 acute triangles.

Case 2. At most one angle of Γ is greater than or equal to $\frac{3\pi}{4}$.

Subcase 2.1. Γ has a pivot in its interior.

Then as in Case 1, Γ_d can be triangulated into at most 52 acute triangles.

Subcase 2.2. Γ has no pivot in its interior.

(a). All of the five angles of Γ are obtuse.

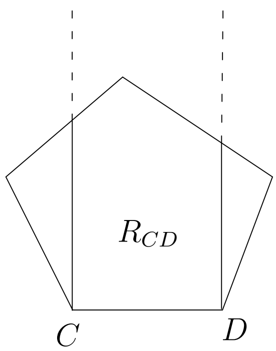


Figure 5.7 Region R_{CD}

For each side of $\Gamma = ABCDE$, say side CD, let $\overline{R}_{CD}$ denote the rectangular region formed by two perpendicular lines of CD starting from C and D. Let $R_{CD} = \Gamma^\circ \cap \overline{R}_{CD}$, as shown in Figure 5.7. Then any point $P \in R_{CD}$ is facing CD in Γ. Let $\mathcal{F} = \{R_{AB}, R_{BC}, R_{CD}, R_{DE}, R_{EA}\}$, then P is a pivot of Γ if and only if $P \in \cap\mathcal{F}$. As a result, Γ has no pivot in its interior means that $\cap\mathcal{F} = \emptyset$. Notice that each member of $\mathcal{F}$ is a convex set, so by Helly's Theorem there are three sides e, f, g of Γ such that $R_e \cap R_f \cap R_g = \emptyset$. Furthermore, it is easy to check that e, f and g are not consecutive. Thus we may assume without loss of generality that $e = AE$, $f = BC$, $g = CD$ and parallelogram $CRST = R_{BC} \cap R_{CD}$ lies to the right of R_{AE}, as shown in Figure 5.8. Recall that the angle at A in Γ is obtuse, so B must lie to the left of R_{AE}, namely, B and T are separated by R_{AE}. Let $HF = R_{AE} \cap l_{BT}$, then $HF \subset \mathrm{relint}BT$.

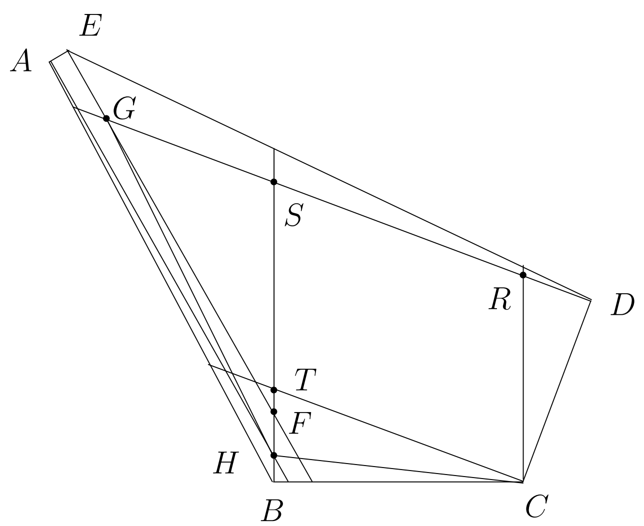


Figure 5.8 Γ has no pivot in its interior

We establish a Cartesian coordinate system with B as origin, BC as $x-$axis and BT as $y-$axis. Let $G = l_{EF} \cap l_{DS}$. The angles of Γ being obtuse, $\angle GFH > \frac{\pi}{2}$ and therefore $\angle AHG = \angle HGF < \frac{\pi}{2}$. So $EAHG$ is a right trapezoid with $\angle EGH > \frac{\pi}{2}$. Notice that $\angle GHC > \angle GBC > \frac{\pi}{2}$, thus $GHCD$ is a quadrilateral with $\angle GHC > \frac{\pi}{2}$, $\angle GDC = \frac{\pi}{2}$. Furthermore, $k_{AH} < 0$, $k_{EG} < 0$ and $k_{GD} < 0$ implies that both $\angle AHB$ and $\angle DGE$ are greater than $\frac{\pi}{2}$.

Now we slightly slide H away from AB in direction perpendicular to AB and slightly slide G in direction $\overrightarrow{EG}$ such that $\angle HAE$, $\angle HBC$, $\angle GDC$ are less than $\frac{\pi}{2}$ while the properties of triangle ABH and triangle DEG are not changed (here the property of a triangle means that both of its acute angles are less than $\frac{\pi}{4}$ or not). Since EAH is an acute triangle, the quadrilateral $EAHG$ can be triangulated into 6 acute triangles such that there is no new vertex introduced on $EG \cup GH$ while there is exactly one new vertex introduced on AH. Similarly, $GHCD$ can be triangulated into 6 acute triangles such that there is no new vertex introduced on $GH \cup CH$ while there is precisely one new vertex introduced on DG. Recall that at most one angle of Γ is greater than or equal to $\frac{3\pi}{4}$, so at most one of the triangle AHB and triangle DEG has an acute angle which is greater than or equal to $\frac{\pi}{4}$. We use Lemma 5.3.2 to triangulate Γ_d into at most $13 \times 2 + 20 + 30 = 76$ non-obtuse triangles, which can be converted into acute triangles by sliding M, M', N, N' if necessary.

(b). Γ has at least one right angle.

Similar to the discussion in (a), we may also assume that $R_{BC} \cap R_{CD} \cap R_{AE} = \emptyset$ and parallelogram $CRST = R_{BC} \cap R_{CD}$ lies to the right of R_{AE}. Then it is easy to deduce that $\angle ABC$, $\angle BCD$ and $\angle DEA$ must be greater than $\frac{\pi}{2}$. Now if $\angle EAB = \frac{\pi}{2}$ (or $\angle CDE = \frac{\pi}{2}$), then we chose a point on l_{BS} (or l_{EF}) which is very close to B (or E) as the point H (or G). The configuration obtained has the same property as that described in Figure 5.8 except that $\angle EAH$ (or $\angle GDC$) is less than $\frac{\pi}{2}$ instead of being equal to $\frac{\pi}{2}$. By a method similar to the one used in (a) we can also triangulate Γ_d into at most 76 acute triangles. □

§5.4 Acute Triangulations of the Double of a Symmetric Smooth Convex set

In this section, we are able to settle the case of the double smooth convex sets Γ_d in case Γ has two perpendicular axes of symmetry.

Theorem 5.4.1. *If Γ is a smooth convex set with two perpendicular symmetry axes, then Γ_d can be triangulated into 72 acute triangles.*

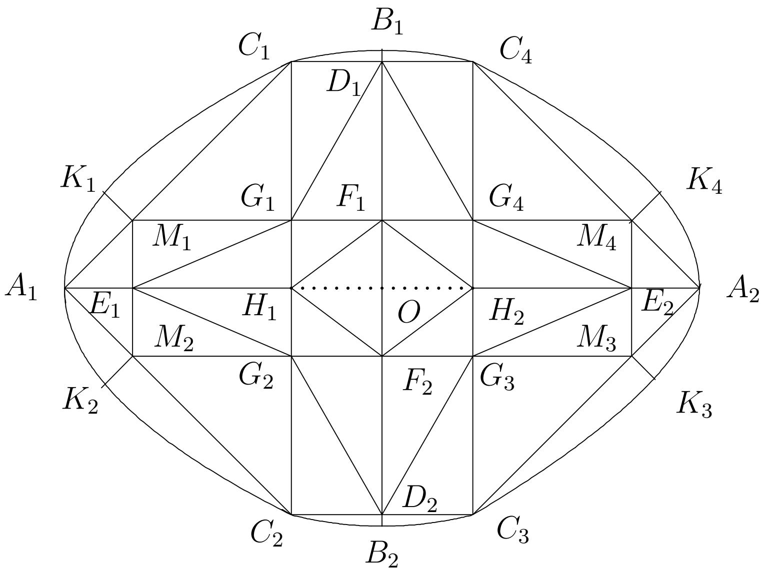


Figure 5.9

Proof. Let $\Gamma \in \mathbb{R}^2$ be a smooth convex set with two perpendicular symmetry axes A_1A_2 and B_1B_2 $(A_1A_2 \cap B_1B_2 = O)$; we may assume without loss of generality that $|A_1A_2| \geq |B_1B_2|$. We denote by $\widehat{A_1B_1}$ (resp. $\widehat{A_1C_1}$) the arc on $bd\Gamma$ from A_1 to B_1 (resp. C_1) not containing A_2.

If $|A_1A_2| > |B_1B_2|$, let $C_1 \in \widehat{A_1B_1}$ satisfying $\angle C_1A_1A_2 = \frac{\pi}{4}$, and slightly slide C_1 on $\widehat{A_1B_1}$ towards A_1. Now let K_1 be the point on $\widehat{A_1C_1}$ which has the maximal distance to A_1C_1 among all the points on $\widehat{A_1C_1}$. Let $M_1 \in \text{relint}A_1C_1$ be the perpendicular foot of K_1. Let $S_{A_1C_1}$ denote the region between $\widehat{A_1C_1}$ and A_1C_1 in Γ. Then $S_{A_1C_1} \cup S'_{A_1C_1}$ can be triangulated into 2 right triangles $A_1M_1M_1'$ and $C_1M_1M_1'$. Let C_2 (resp. K_2, M_2), C_3 (resp. K_3, M_3) and C_4 (resp. K_4, M_4) denote the symmetric points of C_1 (resp. K_1, M_1) according to A_1A_2, O and B_1B_2 respectively, as shown in Figure 5.9. Let $D_1 = B_1B_2 \cap C_1C_4$, $D_2 = B_1B_2 \cap C_2C_3$, $F_1 = B_1B_2 \cap M_1M_4$, $F_2 = B_1B_2 \cap M_2M_3$, $E_1 = A_1A_2 \cap M_1M_2$, $E_2 = A_1A_2 \cap M_3M_4$, $H_1 = A_1A_2 \cap C_1C_2$, $H_2 = A_1A_2 \cap C_3C_4$, $G_1 = C_1C_2 \cap M_1M_4$, $G_2 = C_1C_2 \cap M_2M_3$, $G_3 = C_3C_4 \cap M_2M_3$, $G_4 = C_3C_4 \cap M_1M_4$, and we may assume that $|G_1G_2| \leq |G_3G_4|$. Thus the hexagon $A_1C_2C_3A_2C_4C_1$ can be triangulated into 30 non-obtuse triangles as shown in Figure 5.9. Noticing that $S_{C_1C_4} \cup S'_{C_1C_4}$ can be triangulated into 2 right triangles $C_1D_1D_1'$ and $C_4D_1D_1'$, we can conclude that Γ_d admits a non-obtuse triangulation with size at most 72. Now firstly we slightly slide D_i (resp. F_i) $(i = 1, 2)$ in direction $\overrightarrow{B_iO}$. Secondly we slightly slide E_i (resp. H_i) $(i = 1, 2)$ in direction $\overrightarrow{A_iO}$. Thirdly we slightly slide G_i $(i = 1, 2, 3, 4)$ in direction $\overrightarrow{M_iG_i}$. Finally we slide M_i $(i = 1, 2, 3, 4)$ slightly in direction $\overrightarrow{K_iM_i}$, then we obtain an acute triangulation of Γ_d whose size is at most 72.

If $|A_1A_2| = |B_1B_2|$, let $C_1 \in \widehat{A_1B_1}$ be a point close to B_1. It is easy to check that all the discussion above still holds. □

§5.5 Acute Triangulations of Flat Tori

In this section, we investigate the acute triangulations of the flat tori. It is well-known that any triangulation of the torus contains at least 14 triangles. Concerning acute triangulations, we prove the following result.

Theorem 5.5.1. *Every flat torus can be triangulated into* 16 *acute triangles.*

Proof. There are two cases to consider:

Case 1. There is a (planar) rectangle $A_1A_2A_3A_4$, where A_1, A_2, A_3 and A_4 are identical on T. We may assume that $|A_1A_2| \leq |A_2A_3|$.

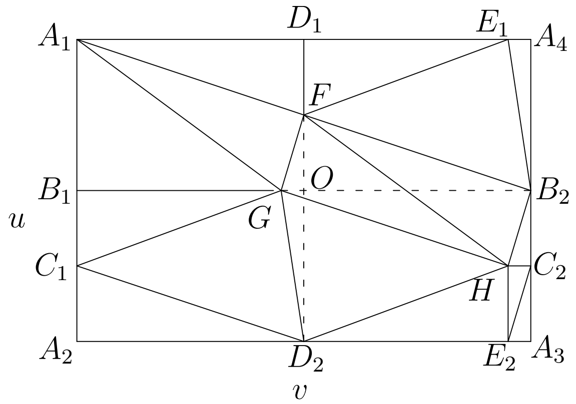


Figure 5.10

Let B_1, B_2, D_1, D_2 be the midpoints of A_1A_2, A_3A_4, A_1A_4, A_2A_3 respectively. Keep in mind that B_i (resp. D_i) $(i = 1, 2)$ are identical on T. Let $A_1B_2 \cap OD_1 = F$, as shown in Fig. 5.10. Let $G \in B_1O$ satisfying $GF \perp A_1B_2$. Let B_2FGH be a rectangle and let C_1, C_2, E_1, E_2 be the orthogonal projections of H on A_1A_2, A_3A_4, A_1A_4, A_2A_3 respectively. Keep in mind again that C_i (resp. E_i) $(i = 1, 2)$ are identical on T. Noticing that $\angle D_2C_1A_2 > \angle GC_1B_1 > \frac{\pi}{4} \Rightarrow \angle GC_1D_2 < \frac{\pi}{2}$, $\angle GD_2O < \angle GFO = \angle FA_1D_1 < \angle FE_1D_1 = \angle HD_2E_2 \Rightarrow \angle GD_2H < \frac{\pi}{2}$ and $\angle FB_2E_1 < \frac{\pi}{2}$, we have $\triangle C_1D_2G \cong \triangle HGD_2 \cong \triangle FB_2E_1$ are acute. It is easy to check that all the edges are segments, i.e., the shortest paths between the corresponding vertices. Then $\mathcal{T}$ admits a non-obtuse triangulation with size 16 as shown in Fig. 5.10.

Now firstly we slightly slide D_2 in direction $\overrightarrow{D_2O}$ such that A_1FD_1, E_1FD_1, $C_1A_2D_2$ and HD_2E_2 become acute. Secondly we slightly slide E_2 in direction $\overrightarrow{E_2A_3}$ such that C_2HE_2 becomes acute. Thirdly we slightly slide H in direction $\overrightarrow{HG}$ such that B_2HC_2 and FHB_2 become acute. Fourthly we slightly slide B_2 in direction $\overrightarrow{B_2O}$ such that A_1B_1G, B_1C_1G and $E_1B_2A_4$ become acute. Fifthly we slightly slide G in direction $\overrightarrow{HG}$ (the sliding distance is much less than that of H) such that A_1GF and FGH become acute. Finally we slightly slide A_3 away from C_2 in direction $\overrightarrow{C_2A_3}$ such that $C_2E_2A_3$ becomes acute. Thus we obtain an acute triangulation of $\mathcal{T}$ with size 16.

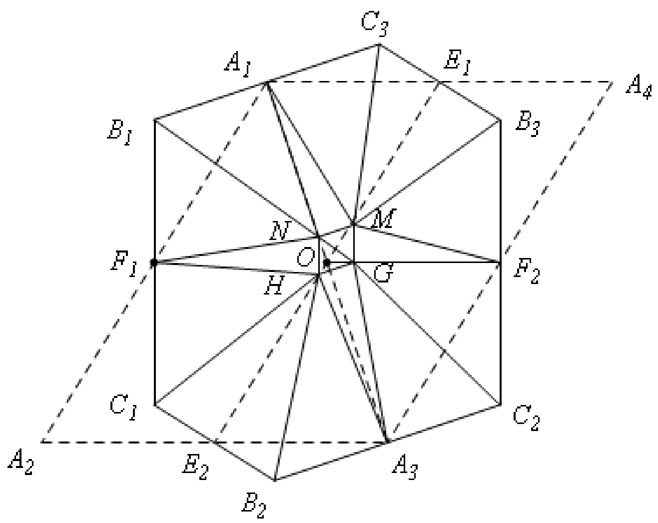


Figure 5.11

Case 2. There is a (planar) parallelogram $A_1A_2A_3A_4$ with center O, where

A_1, A_2, A_3 and A_4 are identical on T. We may assume that $|A_1A_2| \geq |A_2A_3|$.

Now we construct a hexagon on the basis of $A_1A_2A_3A_4$ as shown in Figure 5.11, where $OF_1 \perp B_1C_1$ (F_1 is the midpoint of A_1A_2), $OA_1 \perp B_1C_3$, $OE_1 \perp C_3B_3$ (E_1 is the midpoint of A_1A_4), and B_i, C_i ($i = 1, 2, 3$), E_j, F_j ($j = 1, 2$) are identical on T. Clearly the hexagon $B_1C_1B_2C_2B_3C_3$ is a unfolding figure of T on the plane.

If $|B_1C_3| \geq |B_3C_3|$, then we chose a point M very close to O on the bisector of angle A_1OF_2 . Let N, G be the orthogonal projections of M on OA_1, OF_2 respectively. Let H satisfy $NH \parallel MG$ and $HG \parallel NM$. Then $NHGM$ is a rhombus with acute angle NMG and H lies on the bisector of angle A_3OF_1, which means that HGN_3 is acute. Thus we obtain an non-obtuse triangulation of T with size 16, as shown in Figure 5.11. Clearly, if we slightly slide N in direction $\overrightarrow{MN}$ and G in direction $\overrightarrow{MG}$, then all the triangles become acute.

If $|B_1C_3| < |B_3C_3|$, we can construct an acute triangulation similarly. □

§5.6 Acute Triangulations of Flat Möbius Strips

Let R be a rectangle. We may assume without loss of generality that it has sides 1 and u. If we identify pairs of points symmetric about the center of R and lying on the sides of length 1, then we obtain a Möbius strip, which will be denoted by M_R.

Let $\mathcal{T}$ be an acute triangulation of M_R. A *transversal* of $\mathcal{T}$ is an edge connecting two side vertices of $\mathcal{T}$ and having non-empty intersection with the interior of M_R.

Now let $R = A_1B_1A_2B_2$ with $|A_1B_1| = 1$; then A_1 and A_2 (respectively, B_1 and B_2) are identical in M_R.

Theorem 5.6.1. *If $u \geq \frac{\sqrt{15}}{3}$, then M_R can be triangulated into 5 acute triangles, and no smaller acute triangulation is possible.*

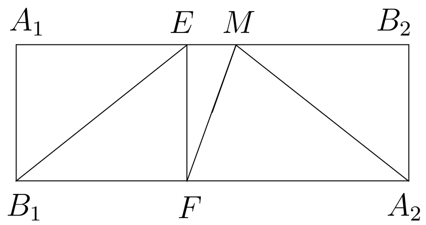


Figure 5.12 $u \geq \sqrt{3}$

Proof. Case 1. $u \geq \sqrt{3}$.

Let $E, M \in A_1B_2$ such that $|B_1E| = |B_2E|$, $|A_1M| = |A_2M|$. Clearly $E \in \text{relint}A_1M$. Let F be the orthogonal projection of E on B_1A_2; then $|A_2F| = |A_2M|$, which implies that MFA_2 is acute. By our construction, it is easy to check that A_1B_1, A_1E, B_1E, B_1F, EM, EF, MB_2, MA_2 and FA_2 are all segments. (Recall that a segment is the shortest path between the corresponding vertices.) Furthermore, since $|MF| = \frac{\sqrt{u^2+1}}{u}$, $|MB_2| = |B_1F| = \frac{u^2-1}{2u}$ and $u \geq \sqrt{3}$, we have $|MF| \leq |MB_2| + |B_1F| < |MA_1| + |A_2F|$, which implies that MF is also a segment. Thus M_R can be triangulated into 5 non-obtuse geodesic triangles, as shown in Fig. 5.12. Now we slightly slide F in direction $\overrightarrow{B_1F}$ and replace the vertex $A_1 = A_2$ of the triangulation with a point on FA_2 close to A_2. So we obtain an acute geodesic triangulation of M_R with size 5.

Case 2. $\frac{\sqrt{15}}{3} \leq u < \sqrt{3}$.

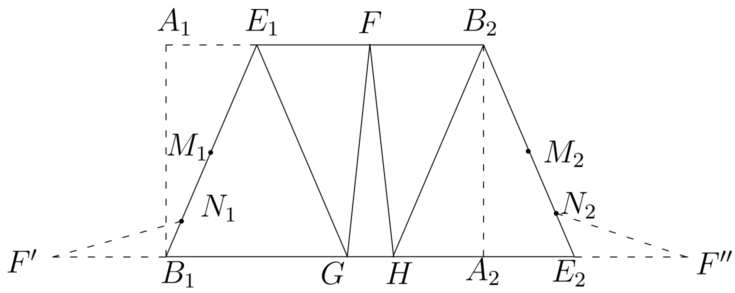


Figure 5.13 $\frac{\sqrt{15}}{3} \leq u < \sqrt{3}$

Let $E_1 \in A_1B_2$ such that $|B_1E_1| = |B_2E_1|$. The Möbius strip M_R is the isosceles trapezoid $E_1B_1E_2B_2$, as shown in Fig. 5.13, where E_1 coincides with E_2 in M_R. Let F be the midpoint of E_1B_2, and let $G, H \in B_1E_2$ such that $|E_1G| = |E_2G|$, $|B_1H| = |B_2H|$. Since $|B_1E_2| - (|B_1G| + |HE_2|) = \frac{3-u^2}{2u} > 0$, $G \in \text{relint} B_1H$. From our construction, it is easy to check that B_1E_1, E_1F, E_1G, B_1G, GH, FB_2, B_2H and HE_2 are all segments. Since $|FG| = \frac{\sqrt{u^4+10u^2+9}}{4u}$, $|FB_2| + |B_1G| = \frac{5u^2-3}{4u}$ and $u \geq \frac{\sqrt{15}}{3}$, we have $|FG| \leq |FB_2| + |B_1G| < |FE_1| + |E_2G|$. Let M_i be the midpoint of E_iB_i $(i = 1, 2)$. M_1, M_2 coincide in M_R. If $N_1 \in B_1M_1$, then $|FB_2|+|B_1G| = |F'B_1|+|B_1G| \leq |F'N_1| + |N_1G|$, where $F' = F$ in M_R (see Figure 5.13). If $N_1 \in E_1M_1$, then $|FE_1|+|E_2G| = |F''E_2|+|E_2G| \leq |F''N_2|+|N_2G|$, where $F'' = F$ and $N_2 = N_1$ in M_R (see Figure 5.13). Hence FG is a segment. Analogously, FH is also a segment. Furthermore, since $\frac{\sqrt{15}}{3} \leq u < \sqrt{3}$, we have $\tan \frac{1}{2}\angle GFH = \frac{3-u^2}{4u} < 1$ and $\tan \frac{1}{2}\angle B_1E_1G = \frac{u^2-1}{2u} < 1$. Thus all the triangles in Fig. 5.13 are acute. Hence we obtain an acute triangulation of M_R with size 5.

Now let $\mathcal{T}$ be an acute triangulation of M_R with t triangles. We regard $\mathcal{T}$ as a planar graph embedded on M_R. If $\mathcal{T}$ has at least one interior vertex, then clearly $t \geq 5$. If $\mathcal{T}$ has no interior vertex, then we assume that it has s side vertices. Notice that every side vertex has degree at least 4, so $s \geq 5$. Now denote by e the number of edges of $\mathcal{T}$. Since $3t + s = 2e = \sum_{x \in V(\mathcal{T})} d(x) \geq 4s$, we have $t \geq s \geq 5$.

The proof is complete. $\square$

Theorem 5.6.2. *If* $1 < u < \frac{\sqrt{15}}{3}$, *then* M_R *can be triangulated into* 8 *acute triangles.*

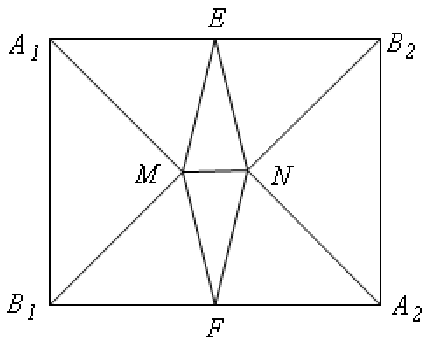


Figure 5.14 $1 < u < \frac{\sqrt{15}}{3}$

Proof. Let E, F be the midpoints of A_1B_2, A_2B_1 respectively. And let M, N be two points inside R such that both A_1B_1M and A_2B_2N are right isosceles triangles. Since $u > 1$, M lies to the left of N. Thus M_R is triangulated into 8 non-obtuse triangles as shown in Figure 5.14. Now we can slightly slide M

in direction $\overrightarrow{MN}$ and N in direction $\overrightarrow{NM}$ such that all the 8 triangles become acute. □

Theorem 5.6.3. *If $u \leq 1$, then M_R admits an acute triangulation with 9 triangles.*

Proof. There are two cases to consider.

Case 1. $u < 1$.

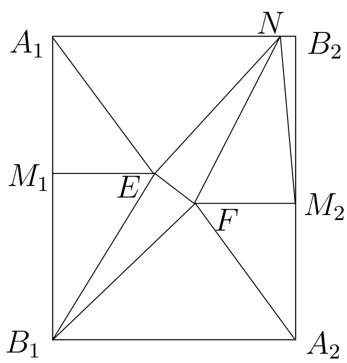


Figure 5.15 $u < 1$

Let $M_1 \in A_1B_1$ be a point which is very close to the midpoint of A_1B_1 and satisfies $|A_1M_1| < |B_1M_1|$ (M_1, M_2 are identical on M_R). Let l denote the perpendicular bisector of B_1B_2. Let E, $F \in l$ such that $M_1E \parallel M_2F \parallel B_1A_2$. Clearly E and F are symmetric with respect to the center of R. Now let $N \in A_1B_2$ be a point very close to B_2 such that $|NM_2| < |NM_1|$ and NEF is an acute triangle. Since $u < 1$, we obtain a non-obtuse triangulation of M_R with size 9, as shown in Figure 5.15. Now we can first slightly slide F in direction $\overrightarrow{M_2A_2}$ and then slightly slide M_1 (or M_2) in direction $\overrightarrow{EM_1}$ such that all the triangles become acute.

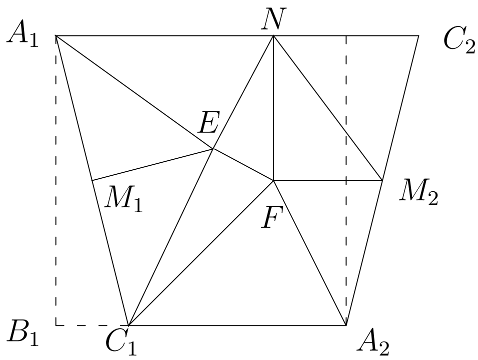


Figure 5.16 $u = 1$

Case 2. $u = 1$.

Let C_1 be the point on B_1A_2 such that $|B_1C_1| = \frac{1}{4}$. The Möbius strip is the isosceles trapezoid $A_1C_1A_2C_2$, where C_1 and C_2 are identical in M_R. Let M_1 be the midpoint of A_1C_1 (M_1 and M_2 are identical in M_R) and let F be the midpoint of C_1C_2. Now denote by N the orthogonal projection of F on A_1C_2. For the sake of convenience, let l^C_{AB} denote the line passing through the point C and perpendicular to AB. Suppose that $l^{M_1}_{A_1C_1} \cap l^F_{C_1N} = E$. Thus M_R admits a geodesic triangulation with size 9, as shown in Figure 5.16. Clearly, the triangles C_1FA_2, FA_2M_2 and NM_2C_2 are acute. Now we establish an x-y coordinate system with C_1 as origin, C_1A_2 as x-axis and $l^{C_1}_{C_1A_2}$ as y-axis. Denote by P the orthogonal projection of F on C_1N. By elementary calculations, we establish that $P = (\frac{3}{10}, \frac{3}{5})$ and $E = (\frac{7}{24}, \frac{29}{48})$, which implies that both NEF and C_1EF are acute. Furthermore, it is easy to check that $|A_1E|^2 + |EN|^2 = \frac{26^2+2\cdot 19^2+10^2}{48^2} > \frac{3^2}{4^2} = |A_1N|^2$, and hence A_1EN is acute. Now firstly we can slightly slide F in direction $\overrightarrow{M_2F}$ such that NFM_2 become acute and C_1F is still a segment. Then we can slightly slide M_1 in direction $\overrightarrow{EM_1}$ such that both A_1M_1E and C_1M_1E become acute.

$\square$

Theorem 5.6.4. *Let $\mathcal{M}$ denote the family of all flat Möbius strips. Then $f(\mathcal{M}) = 9$.*

Proof. Theorem 5.6.1, 5.6.2 and 5.6.3 imply that $f(\mathcal{M}) \leq 9$. Now we shall show that $f(\mathcal{M}) \geq 9$.

Let $\mathcal{T}$ be an acute triangulation of M_R $(u < 1)$ with i interior vertices, s side vertices and t triangles. Trivially $s \geq 3$. Since $u < 1$, there is no transversal in $\mathcal{T}$. Hence $i \geq 1$.

If $i = 1$, then $s \geq 5$. Since each side vertex has degree at least 4, there must be at least one transversal emanating from it, which is impossible.

If $i = 2$, then $s \geq 4$. If $s = 4$, then both interior vertices have degree 5, and all side vertices have degree 4. So we have $3t + 4 = 2e = 26$, which is impossible. If $s \geq 5$, then by $3t + s = 2e \geq 10 + 4s$ we have $3t \geq 10 + 3s \geq 25$, whence $t \geq 9$.

If $i = 3$ and $s = 3$, then all the vertices of $\mathcal{T}$ have degree 5. So $3t + 3 = 2e = 30$ and thus $t = 9$.

If $i = 3$ and $s \geq 4$, then $3t + s = 2e \geq 15 + 4s$ implies that $3t \geq 15 + 3s \geq 27$ and $t \geq 9$.

If $i \geq 4$ and $s \geq 3$, then from $3t + s = 2e \geq 5i + 4s$ we can conclude that $3t \geq 5i + 3s \geq 29$ whence $t > 9$. $\square$

Chapter 6. Acute Triangulations of Rectangles with Angles Bounded Below

Impulses from applied mathematics led to the question of finding triangulations with angles bounded away from π and 0. A very natural upper bound is $\pi/2$, and this is the reason for studying acute or non-obtuse triangulations. We shall consider in this section acute triangulations whose angles are also bounded away from 0. We shall see that this requirement may dramatically increase the number N of necessary triangles. So, there are natural families of polygons, like that of all rectangles, for which no upper bound on N can be given, once all angles must admit a lower bound ε. We are not trying to present an exhaustive study of this kind of triangulations, but pick up the interesting, exemplifying case of rectangles only.

For any triangulation $\mathcal{T}$, let $\delta_{\mathcal{T}}$ denote the minimal value among all the angles in $\mathcal{T}$.

§6.1 Acute Triangulations of Squares

Clearly, any acute triangulation $\mathcal{T}$ of a square satisfies $\delta_{\mathcal{T}} \leq \frac{\pi}{4}$. Moreover, we have the following result.

Theorem 6.1.1. *The square admits an acute triangulation $\mathcal{T}$ such that* $\mathrm{card}\mathcal{T} =$ 14 *and* $\delta_{\mathcal{T}} = \frac{\pi}{4}$.

Proof. For any square $ABCD$, let E, F, G and H be the midpoints of AB, BC, CD and DA respectively. Let $A_1, C_1 \in AC$ satisfying $\angle AHA_1 = \frac{\pi}{3}$,

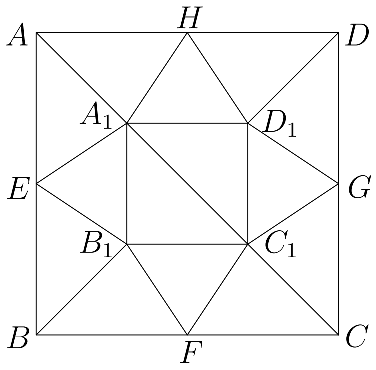


Figure 6.1 card$\mathcal{T} = 14$, $\delta_{\mathcal{T}} = \frac{\pi}{4}$

$\angle CFC_1 = \frac{\pi}{3}$; let B_1, $D_1 \in BD$ satisfying $\angle BFB_1 = \frac{\pi}{3}$, $\angle DHD_1 = \frac{\pi}{3}$. Thus $ABCD$ can be triangulated into 14 non-obtuse triangles as shown in Fig. 6.1. Now we slightly slide A_1 in direction $\overrightarrow{AA_1}$ and C_1 in direction $\overrightarrow{CC_1}$), and obtain an acute triangulation $\mathcal{T}$ with card$\mathcal{T} = 14$ and $\delta_{\mathcal{T}} = \frac{\pi}{4}$. $\square$

Now we regard an acute triangulation $\mathcal{T}$ as a plane graph, that is, a planar graph embedded in the plane. In a graph, the number of those vertices that have degree i is denoted by ν_i.

The following lemma in [34] will be useful.

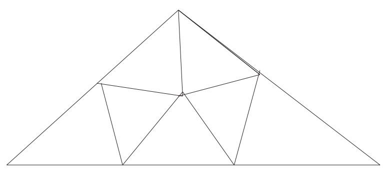


Figure 6.2

Lemma 6.1.2. ([34]) *Let $\mathcal{T}$ be an acute triangulation of a polygon, and suppose that* (1) *$\mathcal{T}$ has a single interior vertex, and* (2) *$\nu_2 + \nu_3 \le 3$, $\nu_2 \le 2$. Then $\mathcal{T}$ is a plane graph isomorphic to the graph shown in Figure 6.2.*

Next we show that the size of the triangulation obtained in Theorem 6.1.1 is best possible.

Theorem 6.1.3. *If $\mathcal{T}$ is an acute triangulation of a square with $\delta_{\mathcal{T}} = \frac{\pi}{4}$, then* $\mathrm{card}\mathcal{T} \geq 14$.

Proof. Let $\mathcal{T}$ be an acute triangulation of a square $\Gamma = ABCD$ (whose center is denoted by O). If $\delta_{\mathcal{T}} = \frac{\pi}{4}$, then there must be precisely one edge emanating from each corner vertex of $\mathcal{T}$, which is colinear with the diagonal of Γ emanating from the same corner. That is to say, $\mathcal{T}$ has at least one interior vertex and each lies on a diagonal of Γ. Let n denote the number of interior vertices of $\mathcal{T}$.

If $n = 1$, then the interior vertex must be O. Since the degree of O is at least 5, there is at least one more edge emanating from O and connecting it with a side vertex. Then the degree of this side vertex will be 3, which is impossible.

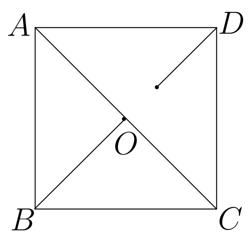


Figure 6.3 $n = 2$

If $n = 2$, then we have the configuration as shown in Fig. 6.3. Apply Lemma 6.1.2 to ACD; then there is one more interior vertex on AC, a contradiction.

If $n = 3$, then there are two possible configurations as shown in Fig. 6.4. By Lemma 6.1.2, Fig. 6.4 (a) is impossible. At the same time, BOC is a right

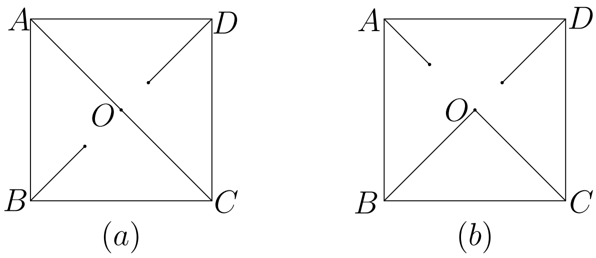


Figure 6.4 $n = 3$

triangle in Fig. 6.4 (b), which can not be converted or triangulated into acute triangles. So Fig. 6.4 (b) is also impossible.

Combining the above observations, we can conclude that $n \geq 4$. Moreover, no side of Γ can be an edge of $\mathcal{T}$. Let m denote the number of side vertices in $\mathcal{T}$; thus $m \geq 4$. So, counting twice in two different ways the edges of T gives $3 \times card\ \mathcal{T} + (m+4) = \sum_{v \in V(\mathcal{T})} d(v) \geq 3 \times 4 + 4m + 5n$, which implies that $\mathrm{card}\mathcal{T} \geq 14$. $\square$

Theorem 6.1.4. *For any $\varepsilon > 0$ there is an acute triangulation $\mathcal{T}$ of the square with* $\mathrm{card}\mathcal{T} = 8$ *and* $\delta_{\mathcal{T}} \geq arctan\frac{16}{63} - \varepsilon$.

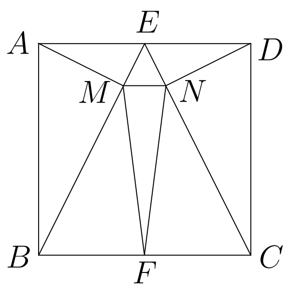


Figure 6.5 $\mathrm{card}\mathcal{T} = 8$, $\delta_{\mathcal{T}} \geq \arctan\frac{16}{63} - \varepsilon$

Proof. Let $ABCD$ be a square and let E, F be the midpoints of AD, BC respectively. Let $M \in BE$, $N \in CE$ such that $AM \perp BE$, $DN \perp CE$. Then

$ABCD$ admits a non-obtuse triangulation $\mathcal{T}'$ (see Fig. 6.5) with card$\mathcal{T}' = 8$ and $\delta_{\mathcal{T}'} = \angle MFN = \arctan\frac{16}{63}$. Now for any $\varepsilon > 0$, we can slightly slide M in direction $\overrightarrow{AM}$ and N in direction $\overrightarrow{DN}$, and obtain an acute triangulation $\mathcal{T}$ such that $\delta_{\mathcal{T}} = \angle MFN \geq \arctan\frac{16}{63} - \varepsilon$. The proof is complete. □

Combining Theorem 6.1.1 and 6.1.4, we have the following corollary.

Corollary 6.1.5. *For any $\theta \in (0, \frac{\pi}{4}]$, every square admits an acute triangulation $\mathcal{T}$ such that $\delta_{\mathcal{T}} \geq \theta$ and*

$$card\mathcal{T} = \begin{cases} 8, & if\ \ \theta \in (0, \arctan\frac{16}{63}); \\ 14, & if\ \ \theta \in [\arctan\frac{16}{63}, \frac{\pi}{4}]. \end{cases}$$

§6.2 Acute triangulations of Rectangles

For the sake of convenience, we may assume without loss of generality that all the rectangles discussed in this section have sides 1 and u ($u > 1$). Let R denote such a rectangle. Clearly, for any acute triangulation $\mathcal{T}$ of R, $\delta_{\mathcal{T}} \leq \frac{\pi}{4}$.

Theorem 6.2.1. *If $u \in (1, 2]$, then for any $\varepsilon > 0$ R admits an acute triangulation $\mathcal{T}$ such that $\delta_{\mathcal{T}} \geq \frac{\pi}{4} - \varepsilon$ and*

$$card\mathcal{T} = \begin{cases} 8, & if\ \ u \in [\sqrt{2}, 2]; \\ 16, & if\ \ u \in (1, \sqrt{2}). \end{cases}$$

Proof. Let $R = ABCD$ be a rectangle with $|AB| = 1$ and $|BC| = u$. Let E, F be the midpoints of AD, BC respectively.

Case 1. $u \in [\sqrt{2}, 2]$.

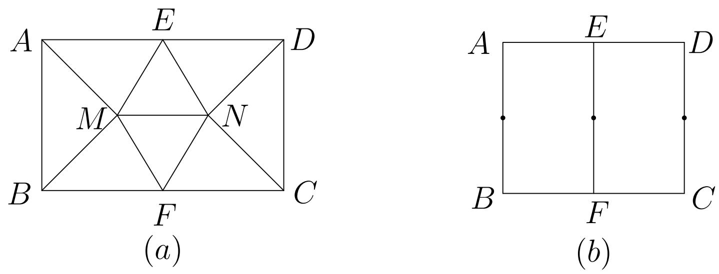


Figure 6.6 $u \in (1, 2]$

Let M be the intersection of the bisectors of corner A and B; let N be the intersection of the bisectors of corner C and D. Then R admits a non-obtuse triangulation $\mathcal{T}'$ such that $\text{card} mathcalT' = 8$ and $\delta_{\mathcal{T}'} = \frac{\pi}{4}$, as shown in Fig. 6.6 (a). Now for any $\varepsilon > 0$, we can slightly slide M in direction $\overrightarrow{MN}$ and N in direction $\overrightarrow{NM}$, and obtain an acute triangulation $\mathcal{T}$ such that $\delta_{\mathcal{T}} \geq \frac{\pi}{4} - \varepsilon$.

Case 2. $u \in (1, \sqrt{2})$.

Firstly we dissect R into 2 congruent rectangles $ABFE$ and $EFCD$. If we denote $\frac{|AB|}{|AE|}$ by v, then $v \in (\sqrt{2}, 2)$. By the proof of Case 1, R admits an acute triangulation $\mathcal{T}$ such that card $\mathcal{T} = 16$ and $\delta_{\mathcal{T}} \geq \frac{\pi}{4} - \varepsilon$. $\square$

For the sake of convenience, we call an acute triangulation described in the proof of Case 1 (resp. Case 2) a *type I* (resp. *type II*) acute triangulation of R. For any real number x, let $[x]$ denote the largest integer not larger than x and let $\{x\} = x - [x]$.

Theorem 6.2.2. *For any $\varepsilon > 0$, a rectangle R with $u > 2$ admits an acute* *triangulation* $\mathcal{T}$ *such that* $\delta_{\mathcal{T}} \geq \frac{\pi}{4} - \varepsilon$ *and*

$$card\ \mathcal{T} = \begin{cases} 4u, & if\ \ u\ is\ even; \\ 4u+4, & if\ \ u\ is\ odd; \\ 8([\frac{u}{2}]+1), & if\ \ \{u\} > 0\ and\ u_0 \in [\sqrt{2}, 2); \\ 16([\frac{u}{2}]+1), & if\ \ \{u\} > 0\ and\ u_0 \in (1, \sqrt{2}), \end{cases}$$

where $u_0 = \frac{u}{[\frac{u}{2}]+1}$.

Proof. If u is an even integer, then R can be dissected into $\frac{u}{2}$ rectangles with sides 1 and 2, and each of them admits a type I acute triangulation. Thus we obtain an acute triangulation $\mathcal{T}$ of R with $\delta_{\mathcal{T}} \geq \frac{\pi}{4} - \varepsilon$ and card$\mathcal{T} = 4u$.

If u is an odd integer, then R can be dissected into $\frac{u-3}{2}$ rectangles with sides 1 and two rectangles with sides 1 and 1.5. Noticing that each of them admits a type I acute triangulation, we obtain an acute triangulation $\mathcal{T}$ of R with $\delta_{\mathcal{T}} \geq \frac{\pi}{4} - \varepsilon$ and card$\mathcal{T} = 4u + 4$.

For any u with $\{u\} > 0$, let $u_0 = \frac{u}{[\frac{u}{2}]+1}$. Then $u_0 \in (1, 2)$. Now we dissect R into $[\frac{u}{2}] + 1$ rectangles with sides 1 and u_0. If $u_0 \in [\sqrt{2}, 2)$ (resp. $u_0 \in (1, \sqrt{2})$), then each of them admits a type I (resp. type II) acute triangulation. Thus we obtain a desirable acute triangulation. □

Lemma 6.2.3. *If* $u \geq 2$*, then* R *admits a non-obtuse triangulation* $\mathcal{T}'$ *such that* card$\mathcal{T}' = 8$ *and* $\delta_{\mathcal{T}'} = \arctan \frac{2}{\sqrt{u^2-4}+u}$*. Furthermore,* $\delta_{\mathcal{T}'}$ *is a continuous decreasing function of* u*.*

Proof. Let $R = ABCD$ be a rectangle with $|AB| = 1$ and $|BC| = u \geq 2$, and let E, F be the midpoints of AD, BC respectively. Suppose that M is the rightmost intersecting points of the two circles with diameter AE, BF and N is the leftmost intersecting points of the two circles with diameter ED,

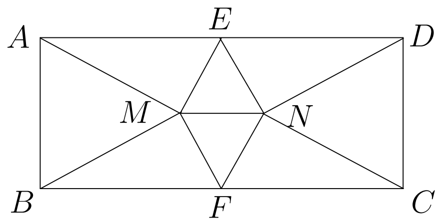


Figure 6.7 A non-obtuse triangulation of R with $u \geq 2$

FC. Since $\triangle AME \cong \triangle BMF \cong \triangle CNF \cong \triangle DNE$, $\triangle AMB \sim \triangle DNC \sim \triangle MEN \sim \triangle MFN$, $\angle AMB = 2\angle MAE$ and $\tan\angle MAE = \frac{2}{\sqrt{u^2-4}+u} \leq 1$, R admits a non-obtuse triangulation $\mathcal{T}'$ with size 8, as shown in Fig. 6.7. So $\delta_{\mathcal{T}'} = \arctan\frac{2}{\sqrt{u^2-4}+u}$ is a continuous decreasing function of u under the condition $u \geq 2$. $\square$

By the proof of Lemma 6.2.3, we can easily obtain the following corollary.

Corollary 6.2.4. *For any $\theta \in (0, \frac{\pi}{4})$, there is a rectangle R with sides 1 and $2(\tan\theta + \frac{1}{\tan\theta})$ which admits a non-obtuse triangulation $\mathcal{T}'$ satisfying card$\mathcal{T}' = 8$ and $\delta_{\mathcal{T}'} = \theta$.*

Theorem 6.2.5. *If $u \geq 2$, then for any $\varepsilon > 0$ there is an acute triangulation $\mathcal{T}$ of R such that* $\mathrm{card}\mathcal{T} = 8$ *and* $\delta_{\mathcal{T}} \geq \arctan\frac{2}{\sqrt{u^2-4}+u} - \varepsilon$.

Proof. By Lemma 6.2.3, R admits a non-obtuse triangulation $\mathcal{T}'$ such that card$\mathcal{T}' = 8$ and $\delta_{\mathcal{T}'} = \arctan\frac{2}{\sqrt{u^2-4}+u}$. For any $\varepsilon > 0$, we can slightly slide M in direction $\overrightarrow{MN}$ and N in direction $\overrightarrow{NM}$, and obtain an acute triangulation $\mathcal{T}$ such that all the angles is not less than $\arctan\frac{2}{\sqrt{u^2-4}+u} - \varepsilon$. This ends the proof. $\square$

Theorem 6.2.6. *If $u > 2$, then for any $\theta \in (0, \frac{\pi}{4})$, R admits an acute triangulation $\mathcal{T}$ such that $\delta_{\mathcal{T}} \geq \theta$ and*

$$card\ \mathcal{T} = \begin{cases} 16([\frac{u}{u_\theta}] + 1), & if\ \overline{u} \in (1, \sqrt{2}); \\ 8([\frac{u}{u_\theta}] + 1), & otherwise, \end{cases}$$

where $u_\theta = 2(\tan\theta + \frac{1}{\tan\theta})$, $\overline{u} = \frac{u}{[\frac{u}{u_\theta}]+1}$.

Proof. By Corollary 6.2.4, for any $\theta \in (0, \frac{\pi}{4})$, there is a rectangle R_θ with sides 1 and $u_\theta = 2(\tan\theta + \frac{1}{\tan\theta}) > 2$ which admits a non-obtuse triangulation $\mathcal{T}'_\theta$ satisfying card$\mathcal{T}'_\theta = 8$ and $\delta_{\mathcal{T}'_\theta} = \theta$. Let $\overline{u} = \frac{u}{[\frac{u}{u_\theta}]+1}$; then clearly $\overline{u} < u_\theta$. Moreover, if $[\frac{u}{u_\theta}] = 0$, then $\overline{u} = u > 2$; if $[\frac{u}{u_\theta}] \geq 1$, then $\overline{u} \geq \frac{u_\theta[\frac{u}{u_\theta}]}{[\frac{u}{u_\theta}]+1} \geq \frac{1}{2}u_\theta > 1$. Hence $1 < \overline{u} < u_\theta$. Now we dissect R into $[\frac{u}{u_\theta}] + 1$ rectangles with sides 1 and $\overline{u}$. For the sake of convenience, we denote such a rectangle by $R_{\overline{u}}$.

If $\overline{u} \in (1, \sqrt{2})$ (resp. $\overline{u} \in [\sqrt{2}, 2]$) , then by the proof of Theorem 6.2.1 we know that $R_{\overline{u}}$ admits an acute triangulation $\mathcal{T}_{R_{\overline{u}}}$ such that card$\mathcal{T}_{R_{\overline{u}}} = 16$ (resp. card$\mathcal{T}_{R_{\overline{u}}} = 8$) and $\delta_{\mathcal{T}_{R_{\overline{u}}}} \geq \theta$. Putting these acute triangulations together, we obtain a desired acute triangulation.

If $\overline{u} \in (2, u_\theta)$, then by Lemma 6.2.3 we know that $R_{\overline{u}}$ admits a non-obtuse triangulation $\mathcal{T}'_{R_{\overline{u}}}$ such that card$\mathcal{T}'_{R_{\overline{u}}} = 8$ and $\delta_{\mathcal{T}'_{R_{\overline{u}}}} > \theta$. Combining all these non-obtuse triangulation together and slightly sliding each interior vertex by the similar methods used in the proof of Theorem 6.2.5, we can obtain an acute triangulation $\mathcal{T}$ of R such that card$\mathcal{T} = 8([\frac{u}{u_\theta}] + 1)$ and $\delta_{\mathcal{T}} \geq \theta$.

The proof is complete. □

Remark. If $\theta \in (0, \arctan(\sqrt{2} - 1)]$, namely, $u_\theta \geq 2\sqrt{2}$, or $[\frac{u}{u_\theta}] \geq 3$, then we always have $\overline{u} \geq \sqrt{2}$. Furthermore, if $\overline{u} > 2$, then the size of the acute triangulation described in Theorem 6.2.6 is best possible.

Recalling that every acute triangulation of any rectangle has size at least 8, we discuss a little bit more on the acute triangulations of rectangles when $u \in (1, \sqrt{2})$.

Theorem 6.2.7. *If $u \in (1, \sqrt{2})$, then for any $\varepsilon > 0$ R admits an acute triangulation $\mathcal{T}$ such that* $\mathrm{card}\mathcal{T} = 8$ *and*

$$\delta_{\mathcal{T}} \geq \begin{cases} \arctan \frac{16u^3}{64-u^3} - \varepsilon, \; if \quad u \in (1, u_0]; \\ \arctan \frac{u}{2} - \varepsilon, \qquad if \;\; u \in ([u_0, \sqrt{2}), \end{cases}$$

where

$$u_0 = \frac{\sqrt{6}[(108+12\sqrt{177})^{\frac{1}{3}}(108+12\sqrt{177})^{\frac{2}{3}} - 24]^{\frac{1}{2}}}{3(108+12\sqrt{177})^{\frac{1}{3}}} = 1.3467...$$

Proof. Let $R = ABCD$ be a rectangle with sides $|AB| = 1$ and $|BC| = u$. Then by the similar method used in the proof of Theorem 6.1.4, R admits a non-obtuse triangulation $\mathcal{T}'$ with size 8, also see Fig. 6.5. Let $\angle MAE = \alpha$, $\angle MFN = \beta$. Then all the rest of angles in $\mathcal{T}'$ are not less than α or β. By calculating we have $\tan \alpha = \frac{u}{2}$, $\tan \beta = \frac{16u^3}{64-u^3}$ and

$$u_0 = \frac{\sqrt{6}[(108+12\sqrt{177})^{\frac{1}{3}}(108+12\sqrt{177})^{\frac{2}{3}} - 24]^{\frac{1}{2}}}{3(108+12\sqrt{177})^{\frac{1}{3}}}$$

is the unique positive real root of $\frac{u}{2} = \frac{16u^3}{64-u^3}$. Hence

$$\delta_{\mathcal{T}'} = \begin{cases} \arctan \frac{16u^3}{64-u^3}, \; if \quad u \in (1, u_0]; \\ \arctan \frac{u}{2}, \qquad if \;\; u \in ([u_0, \sqrt{2}). \end{cases}$$

Now, for any $\varepsilon > 0$, we can slightly slide M in direction $\overrightarrow{AM}$ and N in direction $\overrightarrow{DN}$, and obtain an acute triangulation $\mathcal{T}$ of R such that $\delta_{\mathcal{T}} \geq \delta_{\mathcal{T}'} - \varepsilon$. The proof is complete. $\square$

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