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DOMAIN TRUNCATION METHODS FOR THE WAVE EQUATION IN A HOMOGENIZATION LIMIT

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Abstract. We consider the wave equation $\partial_t^2 v^\varepsilon - \nabla\cdot(a_\varepsilon\nabla)v^\varepsilon = f$ on an unbounded domain Ω_∞ for highly oscillatory coefficients a_ε with the scaling $a_\varepsilon(x) = a(x/\varepsilon)$. We consider settings in which the homogenization process for this equation is well-understood, which means that $v^\varepsilon \to \bar{v}$ holds for the solution $\bar{v}$ of the homogenized problem $\partial_t^2\bar{v} - \nabla\cdot(a_*\nabla)\bar{v} = f$. In this context, domain truncation methods are studied. The goal is to calculate an approximate solution u^ε on a subdomain, say $\Omega_- \subset \Omega_\infty$. We are ready to solve the ε-problem on Ω_-, but we want to solve only homogenized problems on the unbounded domains Ω_∞ or $\Omega_\infty \setminus \bar{\Omega}_-$. The main task is to define transmission conditions at the interface to have small differences $u^\varepsilon - v^\varepsilon$. We present different methods and corresponding $O(\varepsilon)$ error estimates.

Keywords: Wave equation; homogenization; domain truncation

MSC: 35L05, 35B27

1. Introduction

We study the wave equation in a wave-guide geometry: for dimension $d \geq 2$ and a cross section domain $\Gamma_* \subset \mathbb{R}^{d-1}$ we consider the unbounded domain $\Omega_\infty := (-\infty,\infty)\times\Gamma_* \subset \mathbb{R}^d$ and a time interval $(0,T)$. The starting point is a homogenization problem. For a highly oscillatory coefficient field $a_\varepsilon : \Omega_\infty \to \mathbb{R}^{d\times d}$ we consider the wave equation

$$\Box_\varepsilon v^\varepsilon := [\partial_t^2 - \nabla\cdot(a_\varepsilon\nabla)]v^\varepsilon = f \tag{1.1}$$

on $\Omega_\infty \times (0,T)$. We study periodic coefficients $a_\varepsilon(x) := a(x/\varepsilon)$, where $a = a(y)$ is a Y-periodic field on $\mathbb{R}^d$, $Y := [0,1)^d$. The setting will always be chosen in such a way that the homogenization process is justified: The solutions v^ε converge in some sense to a solution $\bar{v}$ of the homogenized problem

$$\Box_*\bar{v} := [\partial_t^2 - \nabla\cdot(a_*\nabla)]\bar{v} = f \tag{1.2}$$

on $\Omega_\infty\times(0,T)$. Regarding homogenization of the wave equation we refer to Theorem 4.3 in [7] and Chapter 2, 3.2 in [4]. The right hand side is always a given function $f : \Omega_\infty \times [0,T] \to \mathbb{R}$ with compact support. Along the lateral boundaries (for $d = 2$ these are 'top' and 'bottom' of the domain), we think of either homogeneous Dirichlet or homogeneous Neumann conditions. To simplify the notation, we choose vanishing initial data and solve (1.1) with the initial conditions $v^\varepsilon(0) = v_0 := 0$ and $\partial_t v^\varepsilon(0) = v_1 := 0$. Equation (1.2) is solved with the same trivial initial data.

Our interest is to study domain truncation schemes in this context. When the solution v^ε has to be determined numerically, one replaces the domain Ω_∞ by a bounded domain; on the bounded domain, the wave equation is discretized. A relevant research question regards the appropriate choice of boundary condition on the

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artificial boundary that is introduced by the domain truncation. The boundary condition has the task that it does not create artificial reflections, therefore the names *non-reflecting boundary condition* and *transparent boundary condition* are used. The topic is interesting even in the time harmonic setting and even for homogeneous coefficients. Exact transparent boundary conditions can be derived for homogeneous coefficients in special geometries and good approximations of exact conditions are available for general geometries (but still homogeneous coefficients).

Our aim is to combine the question of domain truncation with the homogenization limit. In order to fix a setting, we introduce the two domains $\Omega_- := (-\infty, 0) \times \Gamma_*$ and $\Omega_+ := (0, \infty) \times \Gamma_*$. We regard the left domain Ω_- as the domain in which we would like to know the solution and in which we are ready to solve the ε-problem. The artificial boundary is then $\Gamma := \{0\} \times \Gamma_* \subset \Omega_\infty$. We assume that f is compactly supported on $\Omega_- \times [0, T]$. In order to construct a boundary condition on Γ, we are ready to solve the homogenized wave equation on Ω_+. Obviously, for a numerical scheme, the domain truncation must be performed additionally on the left, with another artificial boundary on, say, $\{-L\} \times \Gamma_*$. For notational ease, we restrict ourselves here to the above setting with a truncation only at $\Gamma = \{x_1 = 0\}$.

The schemes of interest have the form that the ε-problem is solved in the left part with a function u^ε,

$$\Box_\varepsilon u^\varepsilon = f \quad \text{in } \Omega_- \times (0, T)\,, \tag{1.3}$$

and the homogenized problem is solved on the right domain with a function u,

$$\Box_* u = 0 \quad \text{in } \Omega_+ \times (0, T)\,, \tag{1.4}$$

both with homogeneous initial data. It remains to choose interface conditions on $\Gamma = \{0\} \times \Gamma_* = \bar{\Omega}_- \cap \bar{\Omega}_+$. To define a complete system of equations, we have to introduce one condition that relates the values of u^ε and u on Γ, and a second condition that relates the fluxes, i.e., derivatives of u^ε and of u on Γ.

We emphasize that the task of defining a numerically useful truncation of the domain can be regarded as solved with a scheme using (1.3)–(1.4), since a truncation of the wave equation with homogeneous coefficients as in (1.4) can be performed with well-established methods.

We will introduce and discuss different choices of interface conditions to complement (1.3)–(1.4). The simplest set of interface conditions is proposed in (2.4)–(2.5). We use the term *half-homogenized problem* below for this set of interface conditions, since they can be understood in a simple way by considering the function w^ε that coincides with u^ε on the left and with u on the right, see (2.3). The interface conditions imply that the function w^ε is of class $H^1(\Omega_\infty)$ and solves a wave equation on Ω_∞, namely one with the coefficients $b_\varepsilon = a_\varepsilon$ on the left and $b_\varepsilon = a_*$ on the right, compare (2.2). The analysis of (2.4)–(2.5) is one of the aims of this article. We derive an error estimate that shows, loosely speaking, $v^\varepsilon - u^\varepsilon = O(\varepsilon)$ on Ω_-. For the precise setting and the precise error estimate, we refer to Theorem 2.1.

We will also suggest two other sets of interface conditions. One is formally of higher order and should therefore provide better approximations. Unfortunately, error estimates for that scheme are only available in the trivial case of ('vertical') laminates, in which the scheme coincides with the half-homogenized scheme. We nevertheless include numerical results for this scheme in Section 5 for a 'horizontal' laminate. These results indeed show a moderate improvement of the results.

A quite different approach is analyzed in Section 4. We do solve (1.3), but we use a boundary condition that is constructed from the solution of a homogenized problem on the entire domain. For this *two-step scheme* we present an analytical error estimate and also numerical results.

Literature and further notation. *Homogenization.* In the 1970s a new field of research within mathematical analysis emerged: The foundations were laid to derive simplified models in homogenization problems [4, 22]. The subject became a huge field, out of the enormous literature we mention just [1] for the introduction of two-scale convergence and [15] for the fully developed theory for stochastic homogenization, regarding porous media applications we mention [6].

As it turned out, the homogenization of the wave equation is somewhat trickier than the homogenization of elliptic or parabolic equations. The problem regards the initial values, which have to be adapted to the coefficient in order to obtain the expected results. General (smooth) initial data must be decomposed into an adapted part and an error part; the latter gives contributions to the homogenization error for all times; in contrast to a parabolic problem, this error is not smeared out by diffusion. The thorough analysis of these results appeared in [7]. Related are surprising features in the long time homogenization of the wave equation, where dispersion effects occur, see [3, 8, 23].

Numerics and domain truncation. Let us turn to the numerical treatment of the wave equation. The wave equation can be discretized with finite differences or finite elements, the resulting finite dimensional problems can be solved numerically. The various aspects of stability and convergence for these schemes is well understood, see [14, 18] and the references therein.

When the wave equation has to be solved on an unbounded domain, one has to truncate the domain. Then one has to impose boundary conditions on the artificial boundary; the aim is to use conditions that do not introduce artificial reflections. The conditions are therefore called non-reflecting or transparent. We do not attempt to describe this vast field here, we mention [12] as one of the early references. Still regarding the time-harmonic case, periodic coefficients in the elliptic operator lead to interesting radiation conditions; we mention [9, 20], and [13] for more general background.

Regarding the time-dependent case, the situation is less satisfactory. We mention the influential papers [10, 16] for typical results for the homogeneous equation. More recent results and numerical aspects can be found in [2, 11, 19, 24]. For the wave equation in, e.g., periodic media, to our knowledge, no transparent boundary condition is known.

Finally, we mention the recent study [21], where truncations of domains in elliptic problems are considered. There, large-scale regularity properties of random elliptic operators are used to construct suitable boundary conditions in the truncated domain.

Our approach in this context. We have the aim to study domain truncations for the wave equation in periodic media. Our approach is to assume that the periodicity ε is small and to make use of homogenization theory.

In all our schemes, we suggest to solve an ε-problem on the truncated (bounded) domain. This, of course, requires a fine numerical resolution on that domain. We furthermore suggest to solve a homogenized wave equation either on the complementary (unbounded) domain, or on the entire domain. We emphasize that this is always possible with little numerical effort: On the one hand, one does not have to use fine grids for the homogeneous problem. On the other hand, one has the possibility to truncate the outer domain again, since transparent boundary conditions for homogeneous coefficients are available. In our experiments, we will actually simply solve the exterior domain problems on sufficiently large domains (with any boundary condition on the newly introduced outer boundary); by finite speed of propagation, this gives accurate solutions.

Notation. In our mathematical results, we restrict to the case of a rectangular cross section $\Gamma_* = (0,1)^{d-1}$. This choice makes it possible to study periodicity conditions along the lateral boundary of the wave guide.

We use summation convention: without writing a summation symbol $\sum$, we mean a summation over every index that occurs twice in an expression. The notation L^p_x is used as a short-hand for L^p-spaces of x-dependent functions, such as $L^p(\Omega)$, and L^p_t as a short-hand for L^p-spaces of t-dependent functions, such as $L^p((0,T))$.

The cross sectional unit cube is $Y' := [0,1)^{d-1}$. Spaces of periodic functions are introduced as $H^1_{\mathrm{per}}(Y) := \{u \in H^1_{\mathrm{loc}}(\mathbb{R}^d) : u(x+e_k) = u(x)$ for a.e. $x \in \mathbb{R}^d$, $\forall k \in \{1,\dots,d\}\}$, the norm is that of $H^1(Y)$. For (macroscopically) periodic functions on Ω_∞ we use $H^1_{\mathrm{per}}(\Omega_\infty) = \{u \in H^1_{\mathrm{loc}}(\mathbb{R}^d) : u(x_1,\cdot) \in H^1_{\mathrm{per}}(Y')$ for a.e. $x_1 \in \mathbb{R}\}$.

Standard elliptic cell solutions are denoted as $\phi_j(y)$, $j = 1,...,d$.

2. Three possible truncation schemes

2.1. Half-homogenized problem.

The half-homogenized problem was already sketched in the introduction. We look for a solution w^ε of the equation

$$\hat{\Box}_\varepsilon w^\varepsilon := [\partial_t^2 - \nabla\cdot(b_\varepsilon\nabla)]w^\varepsilon = f \tag{2.1}$$

on $\Omega_\infty \times (0,T)$, where the coefficient b_ε is given as

$$b_\varepsilon(x) := \begin{cases} a_\varepsilon(x) & \text{for } x \in \Omega_-\,, \\ a_* & \text{for } x \in \Omega_+\,. \end{cases} \tag{2.2}$$

We note that the problem can be equivalently expressed with (1.3) and (1.4). The solution w^ε of the half-homogenized problem (2.1) satisfies

$$w^\varepsilon(x) = \begin{cases} u^\varepsilon(x) & \text{for } x \in \Omega_-\,, \\ u(x) & \text{for } x \in \Omega_+\,, \end{cases} \tag{2.3}$$

when we impose the following interface conditions on Γ (continuity of values and continuity of the flux):

$$u^\varepsilon = u\,, \tag{2.4}$$

$$e_1\cdot a^\varepsilon\nabla u^\varepsilon = e_1\cdot a_*\nabla u\,. \tag{2.5}$$

The main result of this article is the following error estimate for the interface conditions (2.4)–(2.5).

Theorem 2.1 (Error estimate for the half-homogenized problem)**.** *We consider the rectangular cross section $\Gamma_* = (0,1)^{d-1}$ and periodic boundary conditions on the lateral boundary of the wave guide $\Omega_\infty = \mathbb{R}\times\Gamma_*$. Let $a \in C^{0,\alpha}(\mathbb{R}^d)$ be Y-periodic with Hölder exponent $\alpha \in (0,1)$ satisfy the following estimates of uniformly ellipticity with parameter $\lambda > 0$ and boundedness:*

$$\lambda|\xi|^2 \le a(x)\xi\cdot\xi, \qquad |a(x)\xi| \le |\xi| \qquad \text{for all } \xi \in \mathbb{R}^d \text{ and a.e. } x \in \mathbb{R}^d\,. \tag{2.6}$$

There exists $c = c(\alpha, d, \lambda) > 0$ such that the following is true: Let $T > 0$, let $f : \Omega_\infty \times [0,T]$ be sufficiently smooth with compact support, and let $\varepsilon \in (0,1]$ be with $\frac{1}{\varepsilon} \in \mathbb{N}$, let v^ε and w^ε be solutions to (1.1) and (2.1) respectively, both with vanishing initial data. Then, for some $\kappa = \kappa(d,\lambda) > 0$ and the distance $r_\varepsilon = \kappa^{-1}\varepsilon \log(1+\varepsilon^{-1})$, there holds

$$\begin{aligned} &\sup_{t\in[0,T]} \|\nabla v^\varepsilon(t) - \nabla w^\varepsilon(t)\|_{L^2(\Omega_{-,r_\varepsilon})} \\ &\qquad \le c\,\varepsilon\, \left(T(\|\nabla f|_{t=0}\|_{L^2_x} + \|\nabla \partial_t f\|_{L^1_t L^2_x}) + \|f\|_{L^1_t L^2_x} + \|\nabla f\|_{L^1_t L^2_x}\right)\,, \end{aligned} \tag{2.7}$$

where we used slightly reduced domains defined by

$$\Omega_{-,r} := (-\infty, -r) \times \Gamma_*\,.$$

The proof of Theorem 2.1 is given in Section 3. It is based on the observation that both solutions v^ε and w^ε have a distance of order $O(\varepsilon)$ to the homogenized solution $\bar{v}$. As a warning, we mention that, more precisely, the comparison is with an oscillatory limit function that is constructed from $\bar{v}$; the construction uses correctors and is therefore different for v^ε and w^ε. The principal aim is therefore to quantify homogenization errors in the periodic case and in the case of an interface between a periodic and a homogeneous medium. For periodic media, we recover known results; we use a modern language that allows to treat the non-periodic case within the same framework. The non-periodic case is treated with the help of recent estimates for extended correctors by Josien [17].

2.2. **Two-step schemes.** Another scheme determines the solution in two steps; we emphasize that this scheme is not using (1.3)–(1.4). In a first step, we calculate the solution $\bar{v}$ of the homogenized problem. This solution is used to define a boundary condition for u^ε on Γ.

We have two choices: Extracting from $\bar{v}$ Dirichlet data, we can impose a Dirichlet condition for u^ε on Γ. Another choice is to do the same with Neumann data. We will concentrate on the latter and analyze the following scheme:

Calculate the solution $\bar{v}$ of the homogenized problem

$$\Box_* \bar{v} = f \quad \text{in } \Omega_\infty \times (0,T)\,,$$

and use $\bar{v}$ to formulate the following boundary value problem for u^ε: With the correctors ϕ_j of periodic homogenization problems we impose

$$\begin{aligned} \Box_\varepsilon u^\varepsilon &= f && \text{in } \Omega_- \times (0,T)\,,\\ a_\varepsilon \nabla u^\varepsilon \cdot e_1 &= a_\varepsilon(e_i + \nabla\phi_i)\partial_i \bar{v} \cdot e_1 && \text{on } \Gamma \times (0,T)\,. \end{aligned} \tag{2.8}$$

This two-step scheme also allows for $O(\varepsilon)$-estimates. We perform the analysis of the scheme in Section 4.

2.3. **Higher order interface conditions.** It is tempting to improve the interface conditions (2.4)–(2.5) by using an expansion of the solution to a higher order. We treat here only periodic media, i.e., coefficients $a_\varepsilon(x) = a(x/\varepsilon)$ with a Y-periodic field a, and again use the correctors ϕ_j. Given a homogenized solution u on the right domain Ω_+, we expect that a better approximation of v^ε on the right domain is given by

$$u_\varepsilon := u + \varepsilon \partial_j u\, \phi_j\,, \tag{2.9}$$

we recall that summation convention is used throughout this paper. Furthermore, the flux through the interface is $e_1 \cdot a_\varepsilon \nabla v^\varepsilon$. Given u, this flux should be approximated with $e_1 \cdot a_\varepsilon(e_j + \nabla\phi_j)\partial_j u$. We are thus led to the interface conditions

$$u^\varepsilon = u + \varepsilon \partial_j u \, \phi_j \,, \tag{2.10}$$

$$e_1 \cdot a_\varepsilon \nabla u^\varepsilon = e_1 \cdot a_\varepsilon(e_j + \nabla\phi_j)\partial_j u \,. \tag{2.11}$$

We note that, formally, this improves the interface conditions (2.4)–(2.5). While u^ε necessarily has oscillations on scale ε (on the left hand side), with (2.10)–(2.11), we can hope to find u without small scale oscillations.

Our analysis for the scheme with transmission conditions (2.10)–(2.11) did not yield satisfactory results. The only exception is the case of laminates. In the case $a_\varepsilon = a_\varepsilon(x_1)$, the functions ϕ_j vanish for $j \geq 2$ and ϕ_1 is a one-dimensional function, $\phi_1 = \phi_1(y_1)$. We can choose the interface to be at a position with $\phi_1 = 0$. Then (2.10) reduces to (2.4). The right hand side of the second condition is in this case $e_1 \cdot a_\varepsilon(e_j + \nabla\phi_j)\partial_j u = e_1 \cdot a_* \nabla u$, and (2.11) reduces to (2.5). We therefore obtain that the scheme is identical to the half-homogenized scheme for ('vertical') laminates. In particular, our analysis yields error estimates for the scheme in the case of laminates.

We refer to Section 5 for numerical tests with the scheme.

3. Analysis for the half-homogenization scheme

In this section, we provide the proof of Theorem 2.1. We therefore study solutions u^ε of (1.1) and w^ε of (2.1). But before we start the analysis, we have to discuss a quite general assumption.

Assumption 3.1. *Let the coefficient field be given by $a \in L^\infty(\Omega_\infty; \mathbb{R}^{d\times d})$. We consider two properties:*

(i) Uniform ellipticity. *There exists $\lambda > 0$ such that*

$$\lambda|\xi|^2 \leq a(x)\xi\cdot\xi, \qquad |a(x)\xi| \leq |\xi| \qquad \text{for all } \xi \in \mathbb{R}^d \text{ and a.e. } x \in \mathbb{R}^d. \tag{3.1}$$

(ii) Decomposition. *For some a_*, ϕ_i, and σ_i, the following decomposition of a is valid almost everywhere in Ω_∞ and for every $i \leq d$:*

$$a e_i = a_* e_i - a\nabla\phi_i + \nabla\cdot\sigma_i \,. \tag{3.2}$$

Here $a_* \in \mathbb{R}^{d\times d}$ is a single matrix (the homogenized matrix), $\phi_i \in H^1_{\mathrm{per}}(\Omega_\infty)$ (the correctors) are functions that respect the macroscopic periodicity conditions. The correctors satisfy

$$\nabla\cdot a(e_i + \nabla\phi_i) = 0 \,. \tag{3.3}$$

Finally, the functions $\sigma_i \in H^1_{\mathrm{per}}(\Omega_\infty; \mathbb{R}^{d\times d})$ (the flux-correctors) are matrix-valued and respect the macroscopic periodicity conditions. We demand skew-symmetry in the sense $\sigma_{ijk} = -\sigma_{ikj}$. The divergence of σ_i is defined by $(\nabla\cdot\sigma_i)_j = \partial_k\sigma_{ijk}$.

Note that condition (3.3) is equivalent to $0 = \nabla\cdot\nabla\cdot\sigma_i = \partial_j\partial_k\sigma_{ijk}$ and hence a consequence of the skew-symmetry of σ_i.

Remark 3.2. *Assumption 3.1 (ii) is satisfied for Y-periodic coefficient fields a. In this case, ϕ_i is the (unique up to constants) periodic solution of (3.3) in the unit cell*

Y, a_* is obtained by taking the average of (3.2), $a_*e_i = \int_Y a(e_i+\nabla\phi_i)$. In the periodic case, the flux-corrector σ can be constructed such that

$$-\Delta\sigma_{ijk} = \partial_j q_{ik} - \partial_k q_{ij} \quad \text{where} \quad q_i := a(e_i + \nabla\phi_i) - a_*e_i\,. \tag{3.4}$$

We recall the construction in the appendix, see Step 3 in the proof of Lemma 3.4.

For Y-periodic a, the extended corrector (ϕ,σ) are also Y-periodic and, moreover, can be obtained by a rescaling: For $a_\varepsilon = a(\frac{\cdot}{\varepsilon})$ with $\frac{1}{\varepsilon}\in\mathbb{N}$, a decomposition (3.2) is obtained with $\phi_{i,\varepsilon} := \varepsilon\phi_i(\frac{\cdot}{\varepsilon})$ and $\sigma_{i,\varepsilon} := \varepsilon\sigma_i(\frac{\cdot}{\varepsilon})$. The assumption $\frac{1}{\varepsilon}\in\mathbb{N}$ ensures the macroscopic periodicity $\phi_{i,\varepsilon}\in H^1_{\mathrm{per}}(\Omega_\infty)$.

Our first lemma is of a very general nature. We assume the existence of an extended corrector (ϕ,σ) and derive an error estimate in the form of an energy estimate. The method of proof is to multiply the equation with the time derivative of the solution and to integrate. This provides an estimate of the energy of the solution difference in terms of norms of the correctors. In this result, we define an "energy-norm" of a sufficiently smooth function $g:\Omega\times[0,T]\subset\mathbb{R}^d\times\mathbb{R}\to\mathbb{R}$ as

$$E_a(g;\Omega,t) := \left(\frac{1}{2}\int_\Omega |\partial_t g(x,t)|^2 + |\nabla g(x,t)|_a^2\,dx\right)^{\frac{1}{2}}. \tag{3.5}$$

Here, for every x, we set $|\xi|_a = \xi\cdot a(x)\xi$. We suppress the index when the standard Euclidian norm is used, i.e., $E := E_1$.

We emphasize that the coefficient a in the subsequent lemma is the highly oscillatory coefficient a that is treated in Assumption 3.1. In the periodic setting, we would write a_ε instead of a in the lemma below. Correspondingly, we now use the wave operators $\Box := \partial_t^2 - \nabla\cdot a\nabla$ and $\Box_* := \partial_t^2 - \nabla\cdot a_*\nabla$.

Lemma 3.3 (Energy error estimate in homogenization in terms of correctors)**.** *Let Assumption 3.1 be satisfied. There exists $c = c(d,\lambda) > 0$ such that the following is true: For $T>0$ let $f:\Omega_\infty\times[0,T]\to\mathbb{R}$ be sufficiently smooth with compact support. Suppose that v and $\bar v$ satisfy*

$$\Box v = \partial_t^2 v - \nabla\cdot a\nabla v = f\,, \qquad \Box_*\bar v = \partial_t^2\bar v - \nabla\cdot a_*\nabla\bar v = f\,, \tag{3.6}$$

with initial conditions $v_{|t=0} = \bar v_{|t=0} = 0$ and $\partial_t v_{|t=0} = \partial_t\bar v_{|t=0} = 0$. Moreover, suppose $v(\cdot,t)$, $\bar v(\cdot,t)\in \dot H^1_{\mathrm{per}}(\Omega_\infty)$ for all t. Then

$$z := v - (\bar v + \phi_i\partial_i\bar v)$$

satisfies

$$\sup_{t\in[0,T]} E(z;\Omega_\infty,t) \le c\|(\phi,\sigma)\nabla^2\partial_t\bar v\|_{L^1_tL^2_x} + c\||\phi|\nabla\partial_t^2\bar v\|_{L^1_tL^2_x} + c\|(\phi,\sigma)\nabla^2\bar v\|_{L^\infty_tL^2_x}\,, \tag{3.7}$$

where (ϕ,σ) is shorthand for $|\phi_i| + |\sigma_i|$.

Proof. Throughout the proof we write $\lesssim$ if $\le$ holds up to a positive multiplicative constant that depends only on λ and d.

Step 1. We claim that the difference $z = v - (\bar v + \phi_i\partial_i\bar v)$ satisfies

$$\Box z = g \quad \text{with} \quad g = \nabla\cdot((a\phi_i - \sigma_i)\nabla\partial_i\bar v) - \phi_i\partial_t^2\partial_i\bar v\,. \tag{3.8}$$

Indeed, we compute

$$\begin{aligned}\Box z &= f - \Box(\bar v + \phi_i \partial_i \bar v)\\ &= \Box_* \bar v - \Box(\bar v + \phi_i \partial_i \bar v)\\ &= -\nabla\cdot(a_*\nabla\bar v - a(\nabla\bar v + \partial_i\bar v\nabla\phi_i + \phi_i\nabla\partial_i\bar v)) - \phi_i\partial_t^2\partial_i\bar v\\ &= -\nabla\cdot((a_* e_i - a(e_i+\nabla\phi_i))\partial_i\bar v) + \nabla\cdot a\phi_i\nabla\partial_i\bar v - \phi_i\partial_t^2\partial_i\bar v\,.\end{aligned}$$

Using (3.2), we obtain

$$\nabla\cdot((a(e_i+\nabla\phi_i) - a_* e_i)\partial_i\bar v) = \nabla\cdot((\nabla\cdot\sigma_i)\partial_i\bar v) = \nabla\cdot(\sigma_i\nabla\partial_i\bar v)\,,$$

where in the last equality we used $\nabla\cdot(\nabla\cdot\sigma) = 0$ and the skew-symmetry of σ_i. The calculation with indices reads, for every sufficiently smooth function η,

$$\nabla\cdot((\nabla\cdot\sigma_i)\eta) = (\partial_j\partial_k\sigma_{ijk})\eta + (\partial_k\sigma_{ijk})\partial_j\eta = \partial_k(\sigma_{ijk}\partial_j\eta) - \sigma_{ijk}\partial_k\partial_j\eta = \partial_k(\sigma_{ijk}\partial_j\eta)\,.$$

Step 2. Multiplication of equation (3.8) with $\partial_t z$ and integrating yields, using $\phi_i \in H^1_{\mathrm{per}}(\Omega_\infty)$ to see that the integration by parts does not involve boundary terms,

$$\begin{aligned}\frac{d}{dt}E_a(z;\Omega_\infty,t)^2 &= \int_{\Omega_\infty} g(x,t)\partial_t z(x,t)\,dx\\ &= \int_{\Omega_\infty}(\sigma_i - a\phi_i)\nabla\partial_i\bar v(x,t)\cdot\nabla\partial_t z(x,t) - \phi_i\partial_t^2\partial_i\bar v(x,t)\partial_t z(x,t)\,dx\,.\end{aligned}$$

Since $z = \partial_t z = 0$ for $t = 0$, we obtain for every $t > 0$

$$\begin{aligned}E(z;\Omega_\infty,t)^2 &\lesssim \left|\int_0^t\int_{\Omega_\infty}(\sigma_i - a\phi_i)\nabla\partial_i\bar v\cdot\nabla\partial_t z + \phi_i\partial_t^2\partial_i\bar v\partial_t z\,dx\,dt\right|\\ &\le \left|\int_0^t\int_{\Omega_\infty}(\sigma_i - a\phi_i)\nabla\partial_i\partial_t\bar v\cdot\nabla z - \phi_i\partial_t^2\partial_i\bar v\partial_t z\,dx\,dt\right|\\ &\quad + \left|\int_{\Omega_\infty}(\sigma_i - a\phi_i)\nabla\partial_i\bar v(x,t)\cdot\nabla z(x,t)\,dx\right|\\ &\lesssim \int_0^t\int_{\Omega_\infty}|(\phi,\sigma)|(|\nabla^2\partial_t\bar v||\nabla z| + |\partial_t^2\nabla\bar v||\partial_t z|)\\ &\quad + \int_{\Omega_\infty}|(\phi,\sigma)||\nabla^2\bar v(t)||\nabla z(t)|\,.\end{aligned}$$

Using Young's inquality and absorbing the norm of $\nabla z(t)$ into the left hand side, we obtain

$$\begin{aligned}E(z;\Omega_\infty,t)^2 &\lesssim \int_0^t\int_{\Omega_\infty}|(\phi,\sigma)|(|\nabla^2\partial_t\bar v||\nabla z| + |\partial_t^2\nabla\bar v||\partial_t z^\varepsilon|)\\ &\quad + \int_{\Omega_\infty}|(\phi,\sigma)|^2|\nabla^2\bar v(x,t)|^2\,dx\,.\end{aligned}$$

We take the supremum over $t \in [0,T]$ and obtain

$$\begin{aligned}\sup_{0\le t\le T}E(z;\Omega_\infty,t)^2 &\lesssim \sup_{0\le t\le T}E(z,\Omega_\infty,t)(\|(\phi,\sigma)\nabla^2\partial_t\bar v\|_{L^1_tL^2_x} + \||\phi|\nabla\partial_t^2\bar v\|_{L^1_tL^2_x})\\ &\quad + \|(\phi,\sigma)\nabla^2\bar v\|^2_{L^\infty_tL^2_x}\,,\end{aligned}$$

and the claim follows. □

Let us note what the above estimate implies in the case of periodic coefficients. By Remark 3.2, the correctors $\phi_{i,\varepsilon}$ and $\sigma_{i,\varepsilon}$ are of order ε in the periodic case. Therefore, in this case, Lemma 3.3 provides the familiar estimate $\|\nabla(v^\varepsilon - \bar{v})\|_{L^\infty_t L^2_x} \lesssim \varepsilon$ whenever $\bar{v}$ is sufficiently smooth.

Our aim is to estimate a difference of functions v^ε and w^ε that satisfy

$$\partial_t^2 v^\varepsilon - \nabla \cdot a_\varepsilon \nabla u = f \quad \text{and} \quad \partial_t^2 w^\varepsilon - \nabla \cdot b_\varepsilon \nabla w^\varepsilon = f$$

with $a_\varepsilon = a(\frac{\cdot}{\varepsilon})$ and b_ε given by (2.2). The idea is that both solutions v^ε and w^ε converge to the same limit solution $\bar{v}$. Indeed, a_ε and b_ε satisfy the decomposition (3.2) with the same constant coefficient field a_*. We estimate the difference between v^ε and w^ε by appealing twice to Lemma 3.3, using once $a = a_\varepsilon$ and once $a = b_\varepsilon$. The key in the proof is to control the extended correctors of b_ε. The next lemma contains the necessary estimates for the extended corrector of the coefficient field b for $\varepsilon = 1$.

Lemma 3.4 (Extended corrector for b, [17])**.** *Let the coefficient field $a : \mathbb{R}^d \to \mathbb{R}^{d\times d}$ be macroscopically periodic in the sense that $a(x + e_k) = a(x)$ for almost every x and every $k \in \{2, ..., d\}$. For some $\alpha \in (0, 1)$ we assume the regularity $a \in C^{0,\alpha}(\mathbb{R}^d)$. Let Assumption 3.1 be satisfied with ellipticity constant λ, homogenized matrix a_*, and extended correctors (ϕ_i, σ_i). Let the coefficient field b be given as: $b(x) = a(x)$ for $x_1 = e_1 \cdot x < 0$ and $b(x) = a_*$ for $x_1 \geq 0$. Then b can be decomposed as*

$$be_i = a_* e_i - b\nabla\widetilde{\phi}_i + \nabla \cdot \widetilde{\sigma}_i \,,$$

where all $\widetilde{\phi}_i \in W^{1,\infty}(\mathbb{R}^d, \mathbb{R}^d)$, $\widetilde{\sigma}_i \in L^\infty(\mathbb{R}^d; \mathbb{R}^{d\times d\times d}) \cap H^1_{\mathrm{loc}}(\mathbb{R}^d; \mathbb{R}^{d\times d\times d})$ satisfy

$$\nabla \cdot b(e_i + \widetilde{\phi}_i) = 0, \quad \nabla \cdot \widetilde{\sigma}_i = b(e_i + \widetilde{\phi}_i) - a_* e_i \,.$$

There exists $C = C(\alpha, d, \lambda) \in [1, \infty)$ and $\kappa = \kappa(d, \lambda) > 0$ such that

$$|\nabla(\phi_j(x) - \tilde{\phi}_j(x))| \leq C \exp(-\kappa|x_1|) \quad \text{for } x_1 < -1 \,, \tag{3.9}$$

$$|\nabla\tilde{\phi}_j(x)| \leq C \exp(-\kappa|x_1|) \quad \text{for } x_1 > 1 \,. \tag{3.10}$$

As indicated above, Lemma 3.4 is essentially contained in [17], see Propositions 5.3 and 5.4, which cover a more general situation. Since it is not obvious how to translate the statements of [17, Proposition 5.3, 5.4] to the present situation, we provide a proof of Lemma 3.4 in the appendix.

We can prove our main result my combining Lemma 3.3 with Lemma 3.4.

Proof of Theorem 2.1. Throughout the proof we write $\lesssim$ if $\leq$ holds up to a multiplicative constant that depends only on α, d and λ. We use the solution $\bar{v}$ of the homogenized problem $\Box_* \bar{v} = f$ in $\Omega_\infty \times (0, T)$ with periodicity boundary conditions on $\partial\Omega_\infty \times (0, T)$, trivial initial conditions and the operator $\Box_*$ defined through the coefficient a_*.

Step 1. We claim that

$$\sup_{t\in[0,T]} \|\nabla v^\varepsilon(t) - \nabla w^\varepsilon(t)\|_{L^2(\Omega_{-,r})} \lesssim \varepsilon(\|\nabla^2 \partial_t \bar{v}\|_{L^1_t L^2_x} + \|\nabla\partial_t^2 \bar{v}\|_{L^1_t L^2_x} + \|\nabla^2 \bar{v}\|_{L^\infty_t L^2_x}) + \exp(-\kappa \tfrac{r}{\varepsilon})\|\nabla\bar{v}(t)\|_{L^2(\Omega_{-,r})} \,. \tag{3.11}$$

We have to study the extended correctors for a and b. For $\frac{1}{\varepsilon} \in \mathbb{N}$, the coefficients $a_\varepsilon := a(\frac{\cdot}{\varepsilon})$ and $b_\varepsilon := b(\frac{\cdot}{\varepsilon})$ satisfy Assumption 3.1 and the correctors and flux-correctors are given by $\phi_{i,\varepsilon} := \varepsilon\phi_i(\frac{\cdot}{\varepsilon})$, $\phi_{i,\varepsilon} := \varepsilon\sigma_i(\frac{\cdot}{\varepsilon})$ and $\widetilde{\phi}_{i,\varepsilon} := \varepsilon\widetilde{\phi}_i(\frac{\cdot}{\varepsilon})$, $\widetilde{\sigma}_{i,\varepsilon} := \varepsilon\widetilde{\sigma}_i(\frac{\cdot}{\varepsilon})$, respectively. By standard (elliptic) regularity and Lemma 3.4 we have

$$\|(\phi,\sigma)\|_{L^\infty(\mathbb{R}^d)} + \|(\widetilde{\phi},\widetilde{\sigma})\|_{L^\infty(\mathbb{R}^d)} \lesssim 1\,. \tag{3.12}$$

The triangle inequality allows to calculate

$$\begin{aligned}
&\|\nabla v^\varepsilon(t) - \nabla w^\varepsilon(t)\|_{L^2(\Omega_{-,r})}\\
&\quad = \|\nabla(v^\varepsilon - (\bar v - \phi_{i,\varepsilon}\,\partial_i \bar v))(t)\|_{L^2(\Omega_\infty)} + \|\nabla(w^\varepsilon - (\bar v - \widetilde{\phi}_{i,\varepsilon}\,\partial_i \bar v))(t)\|_{L^2(\Omega_\infty)}\\
&\qquad + \|\nabla((\phi_{i,\varepsilon} - \widetilde{\phi}_{i,\varepsilon})\partial_i \bar v)(t)\|_{L^2(\Omega_{-,r})}\,.
\end{aligned}$$

The first two terms on the right-hand side above can be estimated with help of Lemma 3.3, (3.12) and the scaling property of the correctors,

$$\begin{aligned}
&\|\nabla(v^\varepsilon - (\bar v - \phi_{i,\varepsilon}\partial_i \bar v))(t)\|_{L^2(\Omega_\infty)} + \|\nabla(w^\varepsilon - (\bar v - \widetilde{\phi}_{i,\varepsilon}\partial_i \bar v))(t)\|_{L^2(\Omega_\infty)}\\
&\quad \lesssim \varepsilon(\|\nabla^2\partial_t \bar v\|_{L^1_t L^2_x} + \|\nabla\partial_t^2 \bar v\|_{L^1_t L^2_x} + \|\nabla^2 \bar v\|_{L^\infty_t L^2_x})\,.
\end{aligned}$$

In the remaining term we evaluate the derivative of the product and use a triangle inequality. For the first term we use once more (3.12) and the scaling property of the correctors. For the second term, we use (3.9) of Lemma 3.4.

$$\|\nabla((\phi_{i,\varepsilon} - \widetilde{\phi}_{i,\varepsilon})\partial_i \bar v)(t)\|_{L^2(\Omega_{-,r})} \lesssim \varepsilon\|\nabla^2 \bar v(t)\|_{L^2(\Omega_\infty)} + \exp(-\kappa r/\varepsilon)\|\nabla \bar v(t)\|_{L^2(\Omega_{-,r})}\,.$$

This provides (3.11).

Step 2. Standard regularity results for solutions to the homogeneous wave equation are obtained by testing the wave equation with $\partial_t \bar v$ and with $\partial_t \nabla\cdot a_* \nabla \bar v$. Furthermore, one can differentiate the wave equation with respect to t and then test the result with $\partial_t^2 \nabla\cdot a_*\nabla \bar v$. These steps provide

$$\begin{aligned}
&\|\nabla \bar v\|_{L^\infty_t L^2_x} \lesssim \|f\|_{L^1_t L^2_x}\,, \quad \|\nabla^2 \bar v\|_{L^\infty_t L^2_x} \lesssim \|\nabla f\|_{L^1_t L^2_x}\,,\\
&\|\nabla\partial_t^2 \bar v\|_{L^\infty_t L^2_x} + \|\nabla^2\partial_t \bar v\|_{L^\infty_t L^2_x} \lesssim \|\nabla f(0)\|_{L^2} + \|\nabla\partial_t f\|_{L^1_t L^2_x}\,.
\end{aligned}$$

Together with estimate (3.11), they imply

$$\begin{aligned}
&\sup_{t\in[0,T]} \|\nabla v^\varepsilon(t) - \nabla w^\varepsilon(t)\|_{L^2(\Omega_{-,r})}\\
&\quad \lesssim \varepsilon(T(\|\nabla f(0)\|_{L^2_x} + \|\nabla\partial_t f\|_{L^1_t L^2_x}) + \|\nabla f\|_{L^1_t L^2_x}) + \exp(-\kappa\tfrac{r}{\varepsilon})\|f\|_{L^1_t L^2_x}\,.
\end{aligned} \tag{3.13}$$

In the theorem, we use the distance $r = r_\varepsilon = \kappa^{-1}\varepsilon\log(1+\varepsilon^{-1})$. The elementary estimate

$$\exp(-\kappa\tfrac{r}{\varepsilon}) = \exp(-\log(1+\varepsilon^{-1})) \le \frac{1}{1+\varepsilon^{-1}} \le \varepsilon$$

provides (2.7) and concludes the proof of the theorem. □

4. Analysis for the two-step scheme

The two-step scheme has an $O(\varepsilon)$ error estimate just as the half-homogenized scheme. We note that the estimates for the two-step scheme provide the error up to the interface, which is slighty better than in the half-homogenized scheme.

Theorem 4.1 (Error estimate for the two-step scheme)**.** *We consider the cross section $\Gamma_* = (0,1)^{d-1}$ and periodic boundary conditions on the lateral boundary of the wave guide $\Omega_\infty = \mathbb{R}\times\Gamma_*$. Let $a \in C^{0,1}(\mathbb{R}^d)$ be Y-periodic and satisfy the ellipticity condition (2.6) with parameter $\lambda > 0$. For all $T > 0$ there exists $c = c(d,\lambda,\|a\|_{C^{0,1}(Y)},T) > 0$ such that the following is true: Let $f : \Omega_\infty\times[0,T]$ be* *sufficiently smooth with compact support, and let* $\varepsilon \in (0,1]$ *be with* $\frac{1}{\varepsilon} \in \mathbb{N}$*, let* v^ε *and* $\bar v$ *be solutions to* (1.1) *and*

$$\Box_* \bar v = f\,,$$

both with vanishing initial data. Moreover, let u^ε *be the solution to*

$$\begin{aligned} \Box_\varepsilon u^\varepsilon &= f && \text{in } \Omega_- \times (0,T) \\ a_\varepsilon \nabla u^\varepsilon \cdot e_1 &= a_\varepsilon (e_i + \nabla \phi_{i,\varepsilon}) \partial_i \bar v \cdot e_1 && \text{in } \Gamma \times (0,T) \end{aligned} \tag{4.1}$$

with vanishing initial data. Here, $\phi_{i,\varepsilon}$ *denotes the rescaled periodic corrector:* $\phi_{i,\varepsilon} = \varepsilon \phi_i(\frac{\cdot}{\varepsilon})$ *with* $\phi_i \in H^1_{\mathrm{per}}(Y)$ *solving* (3.3) *and* $\int_Y \phi_i \, dx = 0$*. Then*

$$\begin{aligned} &\sup_{t\in[0,T]} \left(\|\partial_t (v^\varepsilon - u^\varepsilon)(t)\|_{L^2(\Omega_-)} + \|\nabla (v^\varepsilon - u^\varepsilon)(t)\|_{L^2(\Omega_-)} \right) \\ &\qquad \le c\,\varepsilon \left(\|f\|_{L^\infty_t W^{1,2}_x} + \|\partial_t f\|_{L^\infty_t W^{2,2}_x} + \|\partial_t^2 f\|_{L^\infty_t L^2_x} \right) . \end{aligned} \tag{4.2}$$

Proof. Throughout the proof we write $\lesssim$ if $\le$ holds up to a multiplicative constant that depends only on d, λ and $\|a\|_{C^{0,1}(Y)}$.

Our aim is to derive an estimate for $z^\varepsilon := v^\varepsilon - u^\varepsilon$. We will obtain an energy type estimate, i.e., an estimate for

$$E_\varepsilon(t) := E_{a_\varepsilon}(z^\varepsilon; \Omega_-, t)\,,$$

where E_{a_ε} is defined in (3.5). Loosely speaking, the energy controls ∇z^ε and $\partial_t z^\varepsilon$ in the space $L^\infty_t L^2_x$.

We make use of the flux-corrector $\sigma_{\varepsilon,i}$. They are given by $\sigma_{i,\varepsilon} = \varepsilon \sigma_i(\frac{\cdot}{\varepsilon})$, where $\sigma_i \in H^1_{\mathrm{per}}(Y, \mathbb{R}^{d\times d\times d})$ satisfies (3.2) and the antisymmetry condition $\sigma_{ijk} = -\sigma_{ikj}$.

Step 1. We claim that the difference z^ε satisfies the following estimate:

$$\begin{aligned} \sup_{t\in[0,T]} E_\varepsilon(t) \lesssim& \|\nabla v^\varepsilon - (e_i + \nabla \phi_{i,\varepsilon})\partial_i \bar v\|_{L^\infty_t L^2_x} + \|\nabla \partial_t v^\varepsilon - (e_i + \nabla\phi_{i,\varepsilon})\partial_i \partial_t \bar v\|_{L^1_t L^2_x} \\ &+ \|\partial_t^2 (v^\varepsilon - \bar v)\|_{L^1_t L^2_x} + \varepsilon(1+T)(\|\nabla^2 \bar v\|_{L^\infty_t W^{1,2}_x} + \|\partial_t \nabla^2 \bar v\|_{L^1_t W^{1,2}_x}) . \end{aligned} \tag{4.3}$$

We will obtain (4.3) by testing the equation $\Box_\varepsilon z^\varepsilon = 0$ with $\partial_t z^\varepsilon$. The wave equation for z^ε is a consequence of (1.1) and (2.8). We exploit the boundary conditions for u^ε and the divergence theorem to calculate

$$\begin{aligned} E_\varepsilon(t)^2 &= \int_0^t \int_\Gamma (\partial_t z^\varepsilon) a_\varepsilon (\nabla v^\varepsilon - \nabla u^\varepsilon)\cdot e_1 \, d\mathcal{H}^{d-1}\, ds \\ &= \int_0^t \int_\Gamma (\partial_t z^\varepsilon) a_\varepsilon (\nabla v^\varepsilon - (e_i + \nabla \phi_{i,\varepsilon})\partial_i \bar v)\cdot e_1 \, d\mathcal{H}^{d-1}\, ds \\ &= \int_0^t \int_{\Omega_-} \nabla \cdot ((\partial_t z^\varepsilon) a_\varepsilon (\nabla v^\varepsilon - (e_i + \nabla\phi_{i,\varepsilon})\partial_i \bar v))\, dx\, ds \\ &= I_1^\varepsilon(t) + I_2^\varepsilon(t)\,, \end{aligned} \tag{4.4}$$

with

$$\begin{aligned} I_1^\varepsilon(t) &:= \int_0^t \int_{\Omega_-} \nabla \partial_t z^\varepsilon \cdot (a_\varepsilon(\nabla v^\varepsilon - (e_i + \nabla \phi_{i,\varepsilon})\partial_i \bar v))\, dx\, ds \\ I_2^\varepsilon(t) &:= \int_0^t \int_{\Omega_-} \partial_t z^\varepsilon \nabla \cdot (a_\varepsilon(\nabla v^\varepsilon - (e_i + \nabla \phi_{i,\varepsilon})\partial_i \bar v))\, dx\, ds\,. \end{aligned}$$

Treatment of I_1^ε. We use an integration by parts in time to rewrite the term as

$$I_1^\varepsilon(t) = \int_{\Omega_-} \nabla z^\varepsilon(t) \cdot a_\varepsilon(\nabla v^\varepsilon(t) - (e_i + \nabla\phi_{i,\varepsilon})\partial_i \bar{v}(t))\, dx$$
$$- \int_0^t \int_{\Omega_-} \nabla z^\varepsilon(s) \cdot a_\varepsilon(\nabla\partial_t v^\varepsilon(s) - (e_i + \nabla\phi_{i,\varepsilon})\partial_i\partial_t \bar{v}(s)))\, dx\, ds\,.$$

With Hölder's inequality we obtain

$$|I_1^\varepsilon(t)| \lesssim E_\varepsilon(t) \|\nabla v^\varepsilon - (e_i + \nabla\phi_{i,\varepsilon})\partial_i \bar{v}\|_{L^2(\Omega_-)}$$
$$+ \int_0^t E_\varepsilon(s) \|\nabla\partial_t v^\varepsilon - (e_i + \nabla\phi_{i,\varepsilon})\partial_i\partial_t \bar{v}\|_{L^2(\Omega_-)}\, ds\,. \tag{4.5}$$

Treatment of I_2^ε. In order to estimate $I_2^\varepsilon(t)$, we use $\Box_\varepsilon v^\varepsilon = f = \Box_* \bar{v}$ in the form

$$\nabla \cdot a_\varepsilon \nabla v^\varepsilon = \partial_t^2 v^\varepsilon - f = \partial_t^2 v^\varepsilon - \partial_t^2 \bar{v} + \nabla \cdot a_* \nabla \bar{v}\,,$$

and rewrite $I_2^\varepsilon(t)$ as

$$I_2^\varepsilon(t) = \int_0^t \int_{\Omega_-} \partial_t z^\varepsilon (\partial_t^2 v^\varepsilon - \partial_t^2 \bar{v})\, dx\, dt + I_3^\varepsilon(t)$$

with

$$I_3^\varepsilon(t) := \int_0^t \int_{\Omega_-} \partial_t z^\varepsilon \nabla \cdot ((a_* e_i - a_\varepsilon(e_i + \nabla\phi_{i,\varepsilon}))\partial_i \bar{v})\, dx\, dt\,. \tag{4.6}$$

Treatment of I_3^ε. We use an integration by parts in time and the identity

$$\nabla \cdot ((a_* e_i - a_\varepsilon(e_i + \nabla\phi_{i,\varepsilon}))g) = \nabla \cdot ((\nabla \cdot \sigma_{i,\varepsilon})g) = \nabla \cdot (\sigma_{i,\varepsilon} \nabla g)$$

with $g = \partial_i \bar{v}$ and $g = \partial_i\partial_t \bar{v}$. The divergence theorem allows to calculate

$$I_3^\varepsilon(t) = \int_{\Omega_-} z^\varepsilon(t) \nabla \cdot (\sigma_{i,\varepsilon}\partial_i \nabla \bar{v}(t))\, dx - \int_0^t \int_{\Omega_-} z^\varepsilon \nabla \cdot (\sigma_{i,\varepsilon}\partial_i\partial_t \nabla \bar{v})\, dx\, ds$$
$$= \int_\Gamma z^\varepsilon(t)(\sigma_{i,\varepsilon}\partial_i \nabla \bar{v}(t)) \cdot e_i\, d\mathcal{H}^{d-1} - \int_{\Omega_-} \sigma_{i,\varepsilon}\partial_i \nabla \bar{v}(t) \cdot \nabla z^\varepsilon(t)\, dx$$
$$- \int_0^t \int_\Gamma z^\varepsilon (\sigma_{i,\varepsilon}\partial_i\partial_t \nabla \bar{v}) \cdot e_i\, d\mathcal{H}^{d-1}\, ds + \int_0^t \int_{\Omega_-} \sigma_{i,\varepsilon}\partial_i \nabla\partial_t \bar{v} \cdot \nabla z^\varepsilon\, dx\, ds\,.$$

We next use the smallness of the flux-corrector in the form $\sup_{x\in\mathbb{R}^d} |\sigma_{i,\varepsilon}(x)| \lesssim \varepsilon$ for all $i = 1, \ldots, d$ (which follows from Remark 3.2 and elliptic regularity using the assumption $a \in C^{0,1}(Y)$), and the trace estimate

$$\|g\|_{L^2(\Gamma)} \lesssim \|g\|_{W^{1,2}(\Omega_-)} \qquad \forall g \in W^{1,2}(\Omega_-)\,.$$

We continue the above calculation with

$$|I_3^\varepsilon(t)| \lesssim \varepsilon \|z^\varepsilon(t)\|_{W^{1,2}(\Omega_-)} \|\nabla^2 \bar{v}(t)\|_{W^{1,2}(\Omega_-)}$$
$$+ \varepsilon \int_0^t \|z^\varepsilon(s)\|_{W^{1,2}(\Omega_-)} \|\partial_t \nabla^2 \bar{v}(s)\|_{W^{1,2}(\Omega_-)}\, ds\,. \tag{4.7}$$

Finally, we use $z^\varepsilon(0) = 0$, which implies

$$\|z^\varepsilon(t)\|_{L^2(\Omega_-)} \le t \sup_{s\in[0,t]} \|\partial_t z^\varepsilon(s)\|_{L^2_x(\Omega_-)}\,,$$

and hence also

$$\|z_\varepsilon(t)\|_{W^{1,2}(\Omega_-)} \le \|\nabla z^\varepsilon(t)\|_{L^2(\Omega_-)} + t \sup_{s\in[0,t]} \|\partial_t z^\varepsilon(s)\|_{L^2(\Omega_-)} \lesssim (1+t) \sup_{s\in[0,t]} E_\varepsilon(s)\,.$$

This estimate allows to conclude from (4.7):

$$\begin{aligned}|I_3^\varepsilon(t)| \lesssim\,& \varepsilon(1+t) \sup_{s\in[0,t]} E_\varepsilon(s)\|\nabla^2\bar v(t)\|_{W^{1,2}(\Omega_-)}\\ &+ \varepsilon \int_0^t (1+s) \sup_{s'\in[0,s]} E_\varepsilon(s')\|\nabla^2\partial_t\bar v\|_{W^{1,2}(\Omega_-)}\,ds\,.\end{aligned} \tag{4.8}$$

Conclusion for $I_2^\varepsilon(t)$ and proof of (4.3)*.* Estimate (4.8) allows to conclude for $I_2^\varepsilon(t)$:

$$\begin{aligned}|I_2^\varepsilon(t)| \lesssim\,& \varepsilon(1+t) \sup_{s\in[0,t]} E_\varepsilon(s)\|\nabla^2\bar v(t)\|_{W^{1,2}(\Omega_-)}\\ &+ \sup_{s\in[0,t]} E_\varepsilon(s) \int_0^t (\|\partial_t^2(v^\varepsilon-\bar v)(s)\|_{L^2(\Omega_-)} + \varepsilon(1+s)\|\nabla^2\partial_t\bar v(s)\|_{W^{1,2}(\Omega_-)}\,ds\,.\end{aligned} \tag{4.9}$$

Estimate (4.3) follows from (4.4) together with (4.5) and (4.9) by taking the supremum over $t\in[0,T]$.

Step 2. Derivation of (4.2) and conclusion of the proof.

From the abstract homogenization estimate in Lemma 3.3 we conclude

$$\|\nabla v^\varepsilon - (e_i+\nabla\phi_{i,\varepsilon})\partial_i\bar v\|_{L_t^\infty L_x^2} \lesssim \varepsilon C(T)(\|\nabla^2\partial_t\bar v\|_{L_t^1 L_x^2} + \|\nabla\partial_t^2\bar v\|_{L_t^1 L_x^2} + \|\nabla^2\bar v\|_{L_t^\infty L_x^2})\,.$$

We note that the term $\phi_{i,\varepsilon}\nabla\partial_i\bar v$ does not appear explicitly, since it is included in the right hand side.

We can also obtain estimates for time derivatives. By a differentiation in time we have

$$\Box_\varepsilon\partial_t v^\varepsilon = \partial_t f = \Box_*\partial_t\bar v\,.$$

Since f vanishes at $t=0$, there holds $\partial_t^2\bar v(0) = \partial_t^2 v^\varepsilon(0) = 0$. Thus, we can apply Lemma 3.3 to the time derivatives with the effect that

$$\begin{aligned}&\|\partial_t^2(v^\varepsilon-\bar v)\|_{L_t^\infty L_x^2} + \|\nabla\partial_t v^\varepsilon - (e_i+\nabla\phi_{i,\varepsilon})\partial_t\partial_i\bar v\|_{L_t^\infty L_x^2}\\ &\lesssim \varepsilon C(T)\left(\|\nabla^2\partial_t^2\bar v\|_{L_t^1 L_x^2} + \|\nabla\partial_t^3\bar v\|_{L_t^1 L_x^2} + \|\partial_t\nabla^2\bar v\|_{L_t^\infty L_x^2} + \|\partial_t\bar v\|_{L_t^\infty L_x^2}\right)\,.\end{aligned}$$

Combining the two estimates gives

$$\begin{aligned}&\|\nabla v^\varepsilon - (e_i+\nabla\phi_{i,\varepsilon})\partial_i\bar v\|_{L_t^\infty L_x^2} + \|\nabla\partial_t v^\varepsilon - (e_i+\nabla\phi_{i,\varepsilon})\partial_t\partial_i\bar v\|_{L_t^\infty L_x^2}\\ &\quad + \|\partial_t^2(v^\varepsilon-\bar v)\|_{L_t^\infty L_x^2}\\ &\lesssim \varepsilon C(T)\left(\|\partial_t^2\nabla\bar v\|_{L_t^1 W_x^{1,2}} + \|\nabla\partial_t^3\bar v\|_{L_t^1 L_x^2} + \|\partial_t\bar v\|_{L_t^\infty W_x^{2,2}} + \|\nabla^2\bar v\|_{L_t^\infty L_x^2}\right)\,.\end{aligned} \tag{4.10}$$

The combination of (4.3) and (4.10) yields

$$\sup_{t\in[0,T]} E_\varepsilon(t) \lesssim \varepsilon C(T)\left(\|\partial_t\bar v\|_{L_t^\infty W_x^{3,2}} + \|\partial_t^2\nabla\bar v\|_{L_t^\infty W_x^{1,2}} + \|\nabla^2\bar v\|_{L_t^\infty L_x^2} + \|\nabla\partial_t^3\bar v\|_{L_t^\infty L_x^2}\right)\,.$$

Estimate (4.2) follows from basic regularity estimates for solutions $\bar v$ of $\Box_*\bar v = f$. □

5. Numerical results

In order to validate the different schemes, we develop two test cases and we compute the L^2-error for each method.

5.1. **One space dimension.** As our first problem, we consider (1.1) in one space dimension with $f = 0$, complemented with the initial conditions

$$v^\epsilon = g = 0 \quad \text{in } \Omega_- \times \{0\}\,, \tag{5.1}$$

$$\partial_t v^\epsilon = h \quad \text{in } \Omega_- \times \{0\}\,, \tag{5.2}$$

with

$$h = \begin{cases} 1 + \cos(\pi x_1) & \text{in } \{-3 \le x_1 \le -1\} \times \{0\}\,, \\ 0 & \text{elsewhere.} \end{cases} \tag{5.3}$$

The coefficient a_ε is given by:

$$a_\varepsilon(x) = \sqrt{2} + \sin(2\pi x_1/\varepsilon)\,, \tag{5.4}$$

such that the corresponding homogenized coefficient is

$$a_* = 1\,. \tag{5.5}$$

To implement the various methods, we discretize equation (1.1) using P1 finite elements both in space and time, with a regular mesh with $\Delta x = 0.01$ and $\Delta t = 0.002$ for all methods. Thanks to the principle of finite propagation of the solution of wave equation, we can simply truncate the domain and apply a homogeneous Neumann boundary condition on a sufficiently large domain in order to calculate a reference solution v^ε.

The various solutions obtained by the different methods are superposed with the reference solution in Figure 1 for $t = 2.5$ and $t = 5$. For the value $\varepsilon = 0.2$, Table 1 lists the errors for the various methods.

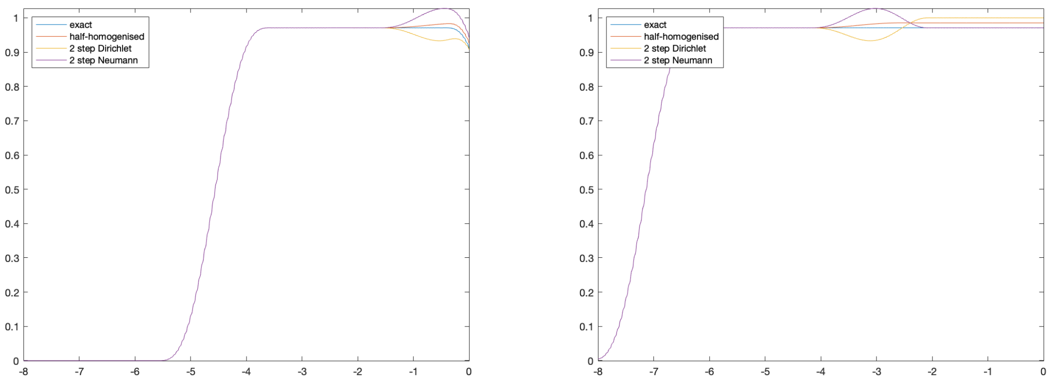

FIGURE 1. One-dimensional test. We plot solutions in Ω_-, the solutions are obtained with different methods. The wave starts to see the artificial boundary at $x = 0$ at time $t = 1$. The left graphs show the solutions at time $T = 2.5$, on the right we see the solutions at time $T = 5$. We see that, in all schemes, the reflection error travels from the artificial boundary to the left. The smallest errors in the max-norm are obtained for the half-homogenized scheme.

Figure 2 shows how fast the error converge as ε tends to zero. The curves show that the error from the two-step Neumann method is less than those of other methods. The error increases when ε becomes too small, since the chosen Δx is not small enough to capture the oscillations of the coefficient field.

Method	$\lVert \cdot - v^\epsilon \rVert$	$\lVert \cdot \rVert$	$\lVert \cdot - \bar{v} \rVert$
v^ϵ	0	2.6030	0.0180
$\bar{v}$	0.0180	2.6064	0
HH	0.0116	2.6090	0.0199
2S - D	0.0206	2.6134	0.0253
2S - N	0.002	2.6038	0.0179

TABLE 1. L^2-norms of the errors at $T = 5$ and for $\varepsilon = 0.2$ in one space dimension. We always evaluate norms in the domain of interest, i.e., in Ω_-. For a comparison, we include differences with the homogenized solution $\bar{v}$.

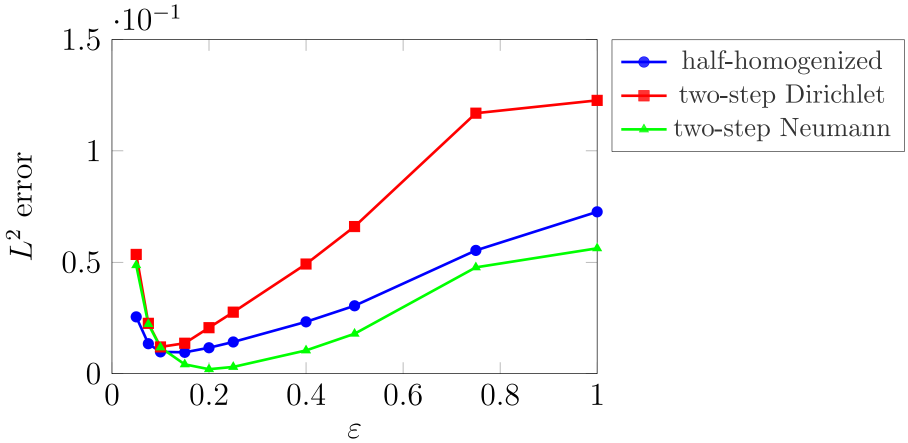

FIGURE 2. L^2-norm of the error at $t = 5$ in one space dimension. The different curves show the different methods.

5.2. **Two space dimensions.** For the 2D case, we use the domain Ω_∞ with the cross section $\Gamma_* = [0, 1]$, and we impose periodic boundary condition on $\partial\Omega_\infty$. As data we use once more $f = 0$ and $g = 0$, and for the initial time derivative h the condition

(5.6)

$$h(x, y) = \begin{cases} \cos\left(2\pi\sqrt{(x+0.5)^2 + (y-0.5)^2}\right) & \text{in } \{(x+0.5)^2 + (y-0.5)^2 \leq 0.0625\}, \\ 0 & \text{else.} \end{cases}$$

In one of our experiments, the coefficient field is given by

$$a_\varepsilon(x_1, x_2) = \begin{pmatrix} \sqrt{2} + \sin(2\pi x_1/\varepsilon) & 0 \\ 0 & \sqrt{2} + \sin(2\pi x_2/\varepsilon) \end{pmatrix},$$

which leads to

$$a_*(x_1, x_2) = \mathrm{Id} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

We discretize the domain with a regular mesh with $\Delta x = 0.02$ and $\Delta t = 0.001$. Figure 3 shows the exact solution v^ε (calculated on the entire domain with the oscillating coefficient) for $\varepsilon = 0.2$.

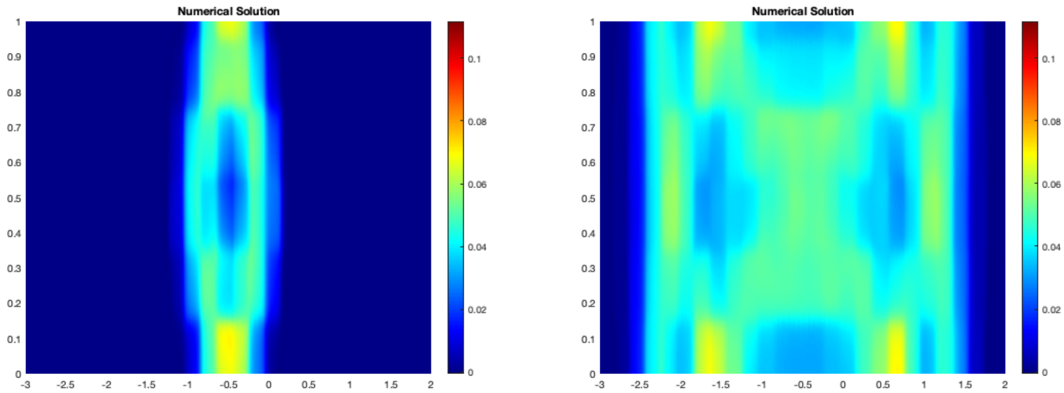

FIGURE 3. The exact solution for $\varepsilon = 0.2$ at time T in two space dimensions. Left: $T = 0.25$. Right: $T = 2$.

Table 2 shows the errors for the different methods for $\varepsilon = 0.2$. The exact solution v^ε and the homogenized solution $\bar{v}$ is computed with a finer discretization, we use $\Delta x = 0.01$. Once more we see that the solutions of our proposed approximate schemes are closer to the exact solution than the homogenized solution.

Method	$\|\cdot - v^\epsilon\|$	$\|\cdot\|$	$\|\cdot - \bar{v}\|$
v^ϵ	0	0.0718	0.0072
$\bar{v}$	0.0072	0.0719	0
HH	0.0028	0.0721	0.0064
2S - D	0.0029	0.0717	0.0066
2S - N	0.0045	0.0728	0.0071

TABLE 2. L^2-norm of errors for different methods. We use $t = 2$ and $\varepsilon = 0.25$ and consider the solution in Ω_-.

Figure 4 the behavior of the error as $\varepsilon \to 0$. In contrast to the 1D experiment, now the half-homogenized method gives the best solution.

We studied additionally another coefficient field. We use the discontinuous and isotropic coefficient a that is given by the scalar field $a(y) = 2$ for $\max\{y_1, y_2\} < 0.5$, and $a(y) = 1$ for $\max\{y_1, y_2\} \geq 0.5$. The results for this coefficient are comparable to the smooth coefficient field, but we now observe a weak performance of the two-step scheme with Neumann conditions. See Figure 5 for the results.

We have performed numerical experiments for various schemes. The half-homogenized method and two variants of the two-step method. We can observe that the half-homogenized scheme provides always a good choice as an approximate system: It is easy to implement and it provides errors that are never significantly larger than the errors of the two-step schemes.

5.3. **Higher order interface conditions.** Even though an analytical treatment did not give satisfactory results, we made numerical tests with the interface conditions (2.10)–(2.11). The tests were performed in the case of a scalar a describing horizontal laminates, which means that the coefficient is $a = a(x_2)$ and the correctors satisfy

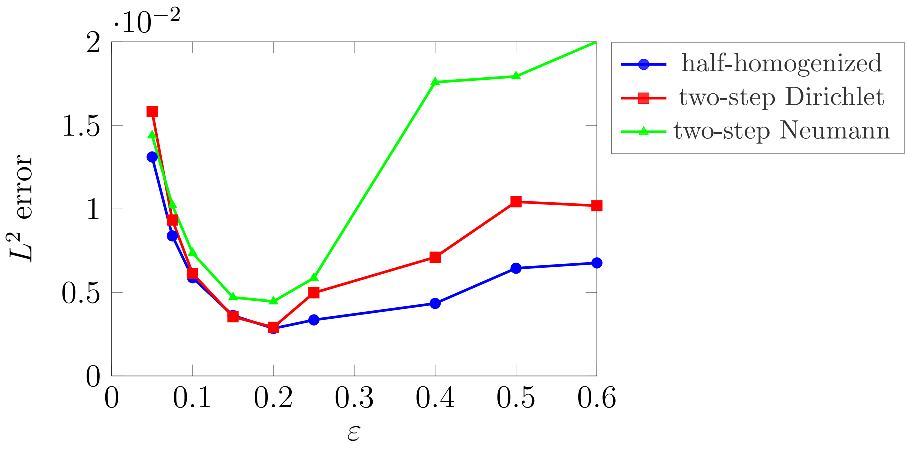

FIGURE 4. L^2 error of the different approximations at $t = 2$ in two space dimensions. We recall that errors are always evaluated in Ω_-.

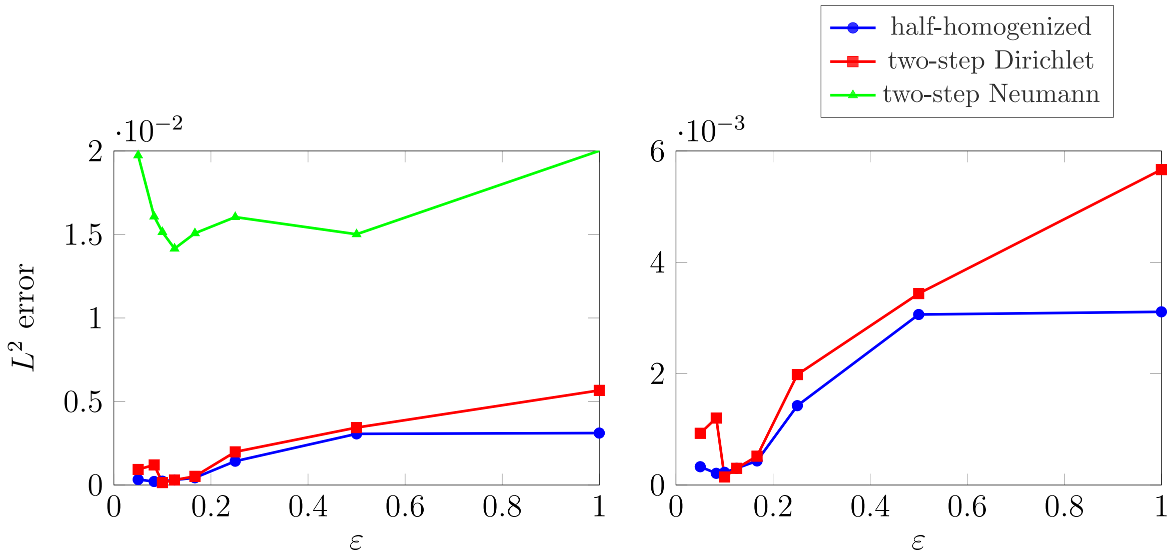

FIGURE 5. $L^2(\Omega_-)$-error in two dimensions for $t = 2$ and an isotropic coefficient a_ε. Here, a discontinuous coefficient field is considered. Left: Comparison of all three methods. We see that the two-step Neumann scheme produces much larger errors than the other two schemes. Right: Zoom of the left figure, showing only the two methods that perform better. We can see again the $O(\varepsilon)$ behavior of the error, at least in the region where the discretization is fine enough to resolve the ε-scale.

$\phi_1 \equiv 0$, $\phi_2 = \phi_2(y_2)$. In this setting, the conditions (2.10)–(2.11) simplify to

$$u^\varepsilon = u + \varepsilon \partial_2 u \, \phi_2 \,, \tag{5.7}$$

$$e_1 \cdot a_\varepsilon \nabla u^\varepsilon = a_\varepsilon \partial_1 u \,. \tag{5.8}$$

Our experiments are with the 1-periodic coefficient ($\varepsilon = 1$) determined by $a(\xi) = 2$ for $\xi \in [0, 1/2)$ and $a(\xi) = 1$ for $\xi \in [1/2, 1)$. In all numerical experiments, we solve the a_ε-wave equation on the domain $\Omega := (0, 4) \times (0, 4)$ and the a_*-wave equation on the domain $R := (4, 6) \times (0, 4)$. The interface is $\Gamma = \{4\} \times (0, 4)$. The two problems are either coupled through the half-homogenized coupling conditions $u^\varepsilon =$

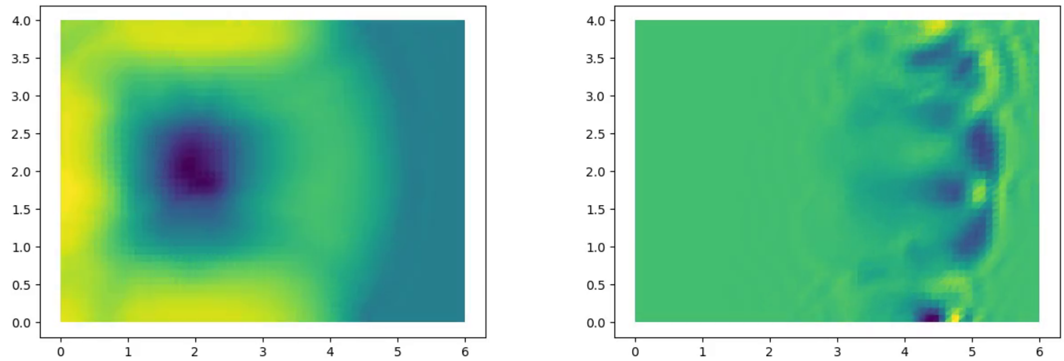

FIGURE 6. Left: A snapshots of the concatenated solution $(u^\varepsilon|u)$ of (5.7)–(5.8); in particular, the correction is included. The solution is shown at some time instance t_0. The wave was generated by continuous initial condition u_0 with support around the point $(2,2)$. Right: For the same time instance t_0, a snapshot of the difference between solution $(u^\varepsilon|u)$ and the true ε-solution v^ε. Errors are large for $x_1 > 4$, since different equations are solved in R. Errors for $x_1 < 4$ are due to incorrect reflections at the artificial interface Γ.

u and $e_1 \cdot a_\varepsilon \nabla u^\varepsilon = a_* \partial_1 u$ along Γ (in this case, we write "without correction"), or by (5.7)–(5.8) ("with correction"); typical solutions are shown in Figure 6. In order to measure solutions and, more importantly, differences of solutions, we use $\|v\|_* := \|v|_\Omega\|_{L^2(\Omega)}$. Here, the "true solution" is the numerical solution v^ε of the a_ε-problem on the combined domain $\Omega \cup \Gamma \cup R$. As a measure for errors we use $\mathrm{err}_\varepsilon = \|u^\varepsilon - v^\varepsilon\|_* / \|v^\varepsilon\|_*$. For results see Figure 7.

We conclude that the numerical tests for higher order interface conditions are promising, at least for horizontal laminates and if one is satisfied with a 30% reduction of the error. The analytical treatment turns out to be more challenging than one might expect.

APPENDIX A. ARGUMENT FOR LEMMA 3.4

As mentioned above, Lemma 3.4 is a special case of [17, Proposition 5.3, 5.4] (see also [5] for related results). We mainly follow the argument of [17], but treat the construction of the flux-corrector σ with special care.

Proof of Lemma 3.4. In order to express the macroscopic periodicity in terms of function spaces, we set

$$L^p_{\mathrm{per}}(\Omega_\infty) := \left\{ u \in L^p_{\mathrm{loc}}(\mathbb{R}^d) \middle| u(x+e_k) = u(x)\, \forall k \in \{2,\dots,d\},\, \forall x \in \mathbb{R}^d \right\} . \tag{A.1}$$

We can define additionally $H^1_{\mathrm{per}}(\Omega_\infty) := H^1_{\mathrm{loc}}(\mathbb{R}^d) \cap L^2_{\mathrm{per}}(\Omega_\infty)$. The norms in these two function spaces are defined with integrals over Ω_∞, i.e., we use the standard norms $\|.\|_{L^p(\Omega_\infty)}$ and $\|.\|_{H^1(\Omega_\infty)}$. When no L^p decay property of the values is demanded, we use the notation with a dot: $\dot{H}^1_{\mathrm{per}}(\Omega_\infty) := \left\{ u \in H^1_{\mathrm{per}}(\Omega_\infty) \middle| \nabla u \in L^2(\Omega_\infty) \right\}$. Throughout the proof we write $\lesssim$ if $\leq$ holds up to a multiplicative constant that depends only on α, d and λ.

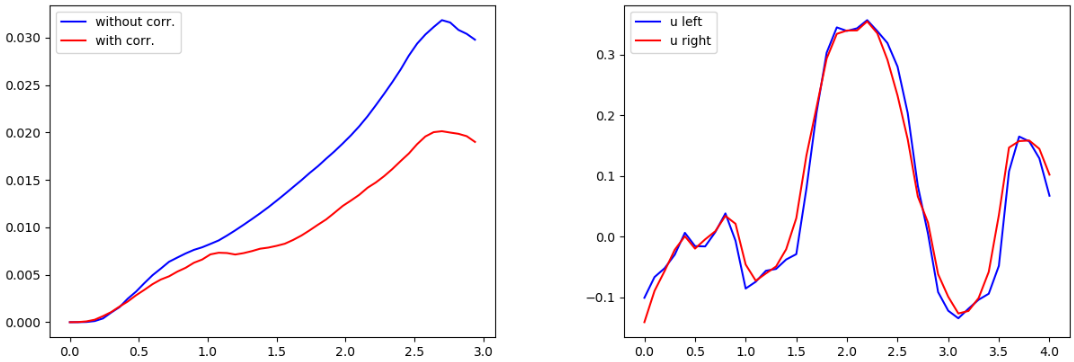

FIGURE 7. Left: The error err_ε over time for the coupled set of wave equations, with and without correction. We observe that the error is descreased by including the correction. The error vanishes up to the time instance when the wave hits the interface at $x_1 = 4$. Right: We consider the solution u^ε with correction. The graphs show the two functions $u^\varepsilon|_\Gamma$ and $u|_\Gamma$. We see that the difference is small, hence the correction in (5.7) is not pronounced. Nevertheless, we can observe that the curve $u|_\Gamma$ seems to be somehow smoother than $u^\varepsilon|_\Gamma$, a result that one can expect: u solves a homogenized problem and should have better regularity than u^ε.

Step 1. Existence. We show the existence of $\widetilde{\phi}_j \in H^1_{\mathrm{per}}(\Omega_\infty)$ satisfying

$$-\nabla \cdot b(\nabla\widetilde{\phi}_j + e_j) = 0\,. \tag{A.2}$$

We choose a smooth cutoff function η_- that depends only on the first coordinate, $\eta_-(x) = \eta_-(x_1)$, with $\eta_- = 1$ on $(-\infty, -1)$ and $\eta_- = 0$ on $(-\frac{1}{2}, \infty)$. A cutoff function η_+ for the complementary domain is defined as $\eta_+(x) := \eta_-(-x)$. We search for $\widetilde{\phi}_j$ in the form

$$\widetilde{\phi}_j = \phi_j \eta_- + z\,. \tag{A.3}$$

This ansatz expresses that $\widetilde{\phi}_j$ is close to ϕ_j in the left part of the domain and small in the right part of the domain. The equation for z can be derived from (A.2) and (A.3) using $\nabla \cdot b_* e_j = 0$:

$$\begin{aligned}
-\nabla\cdot b\nabla z &\overset{(A.3)}{=} -\nabla\cdot\Big[b\nabla(\widetilde{\phi}_j - \phi_j\eta_-)\Big]\\
&= -\nabla\cdot\Big[b(\nabla\widetilde{\phi}_j + e_j) - b(e_j + \eta_-\nabla\phi_j + \phi_j\nabla\eta_-)\Big]\\
&\overset{(A.2)}{=} -\nabla\cdot\left[b_*e_j - b(e_j + \eta_-\nabla\phi_j + \phi_j\nabla\eta_-)\right]\\
&= -\nabla\cdot\left[b_*e_j - b(e_j + \nabla\phi_j)\eta_- - be_j(1-\eta_-) + b\phi_j\nabla\eta_-\right]\\
&= -\nabla\cdot\left[(b_*e_j - b(e_j + \nabla\phi_j))\eta_- + (1-\eta_-)(b_* - b)e_j - b\phi_j\nabla\eta_-\right]\\
&= -\nabla\cdot g - \nabla\cdot f\,,
\end{aligned}$$

with the two functions

$$f := (b_* - b)e_j(1 - \eta_-) - b\phi_j \nabla \eta_- \,,$$
$$g := (a_* e_j - a(e_j + \nabla \phi_j))\eta_- \,.$$

The above formula for g exploits that $b = a$ on the support of η_-. The identity $a_* = b_*$ holds everywhere. It implies, in particular, that the function f has a compact support. As such, by our regularity assumptions, it satisfies $f \in L^p_{\mathrm{per}}(\Omega_\infty)$ for all $p \in [1, \infty]$.

For the subsequent analysis, our aim is to write $-\nabla \cdot b\nabla z$ as the divergence of a function with compact support. Regarding g, this still requires some more calculations. Appealing to (3.2), (3.3), and the skew-symmetry of σ_j, we can write

$$\nabla \cdot g = \nabla \eta_- \cdot (a_* e_j - a(e_j + \nabla \phi_j))$$
$$= \nabla \eta_- \cdot (-\nabla \cdot \sigma_j) = -\nabla \cdot (\sigma_j \nabla \eta) = \nabla \cdot \widetilde{g}$$

for $\widetilde{g} := \sigma_j \nabla \eta$. This brings the equation for z to the final form

$$-\nabla \cdot b\nabla z = -\nabla \cdot h \,, \qquad \text{with} \qquad h := f + \tilde{g} \in L^p_{\mathrm{per}}(\Omega_\infty) \tag{A.4}$$

for every $p \in [1, \infty]$.

In order to construct the corrector, we can now reverse the argument. We define z as the unique Lax-Milgram solution to problem (A.4) and define $\widetilde{\phi}_j$ with the identity (A.3). Then $\widetilde{\phi}_j$ satisfies (A.2).

Step 2. Decay and regularity properties of $\widetilde{\phi}$. We show (3.10), the argument for (3.9) is analogous. We recall that the cross section of $\Omega_\infty = \mathbb{R} \times \Gamma_*$ is given by $\Gamma_* = (0,1)^{d-1}$.

We start with an observation on (A.4). The divergence theorem can be applied on subdomains of the form $\{x \in \Omega_\infty \,|\, r_1 < x_1 < r_2\}$. Together with the fact that h is supported on $\{|x_1| \le 1\}$, it implies that the integral in the subsequent equation (A.5) is independent of r for $|r| > 1$. The property $\nabla z \in L^2(\Omega_\infty)$ implies that the value of the integral vanishes,

$$\int_{\{r\}\times\Gamma_*} b\nabla z \cdot e_1 \, dx = 0 \qquad \forall |r| > 1 \,. \tag{A.5}$$

Multiplication of (A.4) with z and integrating over $\{x \in \Omega_\infty \,|\, R < x_1 < R'\}$ for $1 < R < R'$, exploiting once more that h is supported on $\{|x_1| \le 1\}$, we find

$$\int_{(R,R')\times\Gamma_*} b\nabla z \cdot \nabla z \, dx = \int_{\{R'\}\times\Gamma_*} z b\nabla z \cdot e_1 \, dx - \int_{\{R\}\times\Gamma_*} z b\nabla z \cdot e_1 \, dx \,.$$

Observation (A.5) allows to modify this equation by subtracting averages $z^*_R := \fint_{\{R\}\times\Gamma_*} z$. We obtain

$$\int_{(R,R')\times\Gamma_*} b\nabla z \cdot \nabla z \, dx = \int_{\{R'\}\times\Gamma_*} (z - z^*_{R'}) b\nabla z \cdot e_1 \, dx - \int_{\{R\}\times\Gamma_*} (z - z^*_R) b\nabla z \cdot e_1 \, dx \,.$$

The Poincaré inequality in the $(d-1)$-dimensional slices allows to conclude

$$\int_{(R,R')\times\Gamma_*} b\nabla z \cdot \nabla z \, dx \lesssim \int_{\{R'\}\times\Gamma_*} |\nabla z|^2 \, dx + \int_{\{R\}\times\Gamma_*} |\nabla z|^2 \, dx \,. \tag{A.6}$$

Sending $R' \to \infty$ in the above estimate, using once more $\nabla z \in L^2(\Omega_\infty)$, we obtain

$$F(R) := \int_{(R,\infty)\times\Gamma_*} |\nabla z|^2\, dx \lesssim \int_{\{R\}\times\Gamma_*} |\nabla z|^2\, dx \quad \forall R > 1\,.$$

This implies the differential inequality $F(R) \le -\kappa F'(R)$ for all $R > 1$ and some $\kappa = \kappa(d, \lambda) > 0$. We rewrite this as $\frac{d}{dR}\log(F(R)) = \frac{F'(R)}{F(R)} \le -\kappa$ and thus, for all $R > 1$,

$$\log(F(R)) - \log(F(1)) \le -\kappa(R-1)\,, \quad \text{which yields} \quad F(R) \le F(1)\exp(\kappa)\exp(-\kappa R)\,.$$

Appealing to local regularity theory, we obtain for $x_1 > 1$

$$|\nabla z(x)| \lesssim \exp(-\kappa x_1)\,. \tag{A.7}$$

We have thus deduced (3.10) from (A.4).

Step 3. Construction of $\widetilde{\sigma}$. Let us first recall the construction of σ, i.e., the construction for periodic coefficients a. One seeks σ in the form

$$\sigma_{ijk} = \partial_j N_{ik} - \partial_k N_{ij}\,, \tag{A.8}$$

where N is the solution of class $N \in H^1_{\mathrm{per}}(Y, \mathbb{R}^{d\times d})$ of the elliptic problem

$$-\Delta N_{ij} = (a(e_i + \nabla\phi_i) - a_* e_i)\cdot e_j =: q_{ij}\,. \tag{A.9}$$

The ansatz (A.8) already implies the skew-symmetry $\sigma_{ijk} = -\sigma_{ikj}$. Furthermore, we can calculate in the sense of distributions

$$-\Delta(\partial_k\sigma_{ijk} - q_{ij}) = -(\partial_j\partial_k\Delta N_{ik} - \Delta\Delta N_{ij} - \Delta q_{ij}) = \partial_j\partial_k q_{ik} = 0\,, \tag{A.10}$$

where we used first (A.9) and then the corrector equation in the form $\partial_k q_{ik} = \nabla\cdot a(e_i + \nabla\phi_i) = 0$. Combining Weyl's Lemma with the computations of (A.10), we deduce that $\partial_k\sigma_{ijk} - q_{ij}$ is smooth and harmonic. Since, by construction, $\partial_k\sigma_{ijk} - q_{ij}$ is periodic with vanishing average $\int_Y \partial_k\sigma_{ijk} - q_{ij} = 0$, we obtain $\partial_k\sigma_{ijk} = q_{ij}$. This is the desired property of the corrector σ, compare (3.2).

We now turn to the construction of $\widetilde{\sigma}$ for the coefficient b. Our goal is to construct $\widetilde{N}$ satisfying

$$-\Delta\widetilde{N}_{ij} = (b(e_i + \nabla\widetilde{\phi}_i) - b_* e_i)\cdot e_j =: \widetilde{q}_{ij}\,. \tag{A.11}$$

We do not seek a Y-periodic solution, but we do impose the macroscopic periodicity conditions and demand $\widetilde{N} \in H^2_{\mathrm{per}}(\Omega_\infty; \mathbb{R}^{d\times d})$. Furthermore, we require the boundedness

$$\forall\alpha \in (0,1): \quad \sup_{x\in\mathbb{R}^d} \|\nabla\widetilde{N}_{ij}\|_{C^{0,\alpha}(x)} \le c(\alpha, d, \lambda) < \infty\,. \tag{A.12}$$

Once we have constructed $\widetilde{N}$, we can define $\widetilde{\sigma} \in H^1_{\mathrm{per}}(\Omega_\infty; \mathbb{R}^{d\times d\times d})$ as in the periodic case as

$$\widetilde{\sigma}_{ijk} := \partial_j\widetilde{N}_{ik} - \partial_k\widetilde{N}_{ij}\,.$$

Let us first argue that this construction indeed provides $\widetilde{\sigma}$ with the desired properties. As in the periodic case, we find that $\partial_k\widetilde{\sigma}_{ijk} - \widetilde{q}_{ij}$ is harmonic and smooth. The function has is 1-periodic in every direction $k \in \{2, ..., d\}$ because of the macroscopic periodicity of $\widetilde{N}$ and $\widetilde{\phi}$. In the positive x_1-direction (i.e., direction of e_1, $k = 1$) we observe that $\widetilde{q}_{ij}$ has exponential decay by (A.7) and $\partial_k\widetilde{\sigma}_{ijk}$ is bounded by (A.12). In the negative x_1-direction, $\widetilde{q}_{ij}$ has exponential decay towards a periodic solution and $\partial_k\widetilde{\sigma}_{ijk}$ is bounded by (A.12). All these facts together imply that $\partial_k\widetilde{\sigma}_{ijk} - \widetilde{q}_{ij}$ is bounded. Liouville's theorem implies that this harmonic function must be constant. The exponential decay of $\widetilde{q}_{ij}$ and boundedness of $\widetilde{\sigma}_{ijk}$ for $x_1 \to +\infty$ implies that the constant is 0. We therefore find that $\partial_k \widetilde{\sigma}_{ijk} = \widetilde{q}_{ij}$.

It remains to construct $\widetilde{N}$. We make the following ansatz:

$$\widetilde{N} = \eta_- N + V \,. \tag{A.13}$$

The equation for V reads

$$-\Delta V_{ij} = -\,\Delta \widetilde{N}_{ij} + \eta_- \Delta N_{ij} + 2\nabla \eta_- \cdot \nabla N_{ij} + \Delta \eta_- N_{ij} =: g_{ij} \,. \tag{A.14}$$

We observe that the last two terms have bounded support in x_1 since they contain derivatives of η_-. The first two terms can be calculated using (A.9), (A.11), and $a = b$ on $\{\eta_- \neq 0\}$:

$$-\Delta \widetilde{N}_{ij} + \eta_- \Delta N_{ij} = (1-\eta_-)(b(e_i + \nabla \widetilde{\phi}_i) - b_* e_i) \cdot e_j + \eta_- b(\nabla \widetilde{\phi}_i - \nabla \phi_i) \cdot e_j \,.$$

This implies $g_{ij} \in L^p_{\mathrm{per}}(\Omega_\infty)$ for all $p \in [1,\infty]$; we use here the exponential decay properties (3.9) and (3.10), and the fact that $b = b_*$ on $\{x_1 > 0\}$.

The L^p-property of g allows to find $V \in H^2_{\mathrm{per}}(\Omega_\infty)$ solving (A.14) and satisfying $\sup_{x\in\mathbb{R}^d} \|\nabla V_{ij}\|_{C^{0,\alpha}(x)} \le c(\alpha, d, \lambda) < \infty$; this is shown in the next step of this proof. Having V with these properties, $\widetilde{N}$ can be defined via (A.13), it satisfies (A.11) and (A.12), and we have thus constructed $\widetilde{\sigma}_{ijk}$.

Step 4. We claim the following: For given $g \in L^1_{\mathrm{per}}(\Omega_\infty) \cap L^\infty_{\mathrm{per}}(\Omega_\infty)$ there exists $v \in H^1_{\mathrm{per,loc}}(\Omega_\infty)$ satisfying

$$-\,\Delta v = g \quad \text{and} \quad \sup_{x\in\mathbb{R}^d} \|\nabla v\|_{C^{0,\alpha}(B_1(x))} \lesssim \|g\|_{L^1(\Omega_\infty)} + \|g\|_{L^\infty(\Omega_\infty)} \,. \tag{A.15}$$

Case 1. Suppose that g has vanishing averages in cross sections in the sense that

$$\int_{\Gamma_*} g(x_1, x')\, dx' = 0 \qquad \forall x_1 \in \mathbb{R} \,. \tag{A.16}$$

In this case, g can be written as a divergence. Indeed, for every $x_1 \in \mathbb{R}$, there exists a function $w_{x_1} \in H^1_{\mathrm{per}}(\Gamma_*; \mathbb{R})$ such that

$$-\Delta' w_{x_1} = g(x_1, \cdot) \quad \text{in } \Gamma_* \quad \text{and} \quad \|\nabla' w_{x_1}\|_{L^2(\Gamma_*)} \lesssim \|g(x_1,\cdot)\|_{L^2(\Gamma_*)} \,,$$

where we used the operators $\nabla' = (\partial_2, \partial_3, \dots, \partial_d)$ and $\Delta' = \nabla' \cdot \nabla'$. With the choice $f(x_1, .) := (\,0\,, \nabla' w_{x_1}(.))$, there holds $f \in L^2_{\mathrm{per}}(\Omega_\infty; \mathbb{R}^d)$ and

$$-\,\nabla \cdot f = g \quad \text{and} \quad \|f\|_{L^2(\Omega_\infty)} \lesssim \|g\|_{L^2(\Omega_\infty)} \,. \tag{A.17}$$

Using f, the function v is defined as the unique Lax-Milgram solution $v \in \dot{H}^1_{\mathrm{per}}(\Omega_\infty)$ of

$$-\Delta v = -\nabla \cdot f \,.$$

It satisfies $\|\nabla v\|_{L^2(\Omega_\infty)} \lesssim \|f\|_{L^2(\Omega_\infty)} \lesssim \|g\|_{L^2(\Omega_\infty)}$. Standard elliptic regularity implies (A.15).

Case 2. The general case. We decompose the g as

$$g = g_1 + g_2 \quad \text{where} \quad g_1 = g_1(x_1) := \int_{\Gamma_*} g(x_1, x')\, dx' \,.$$

Appealing to the first case, we find v_2 satisfying (A.15) with g replaced by g_2. The (one dimensional) function

$$v_1(x) := v_1(x_1) := -\int_0^{x_1}\int_0^s g_1(h)\,dh\,ds\,,$$

satisfies (A.15) with g replaced by g_1. Thus $v = v_1 + v_2$ satisfies (A.15). □

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